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Second kind integral equations for the classical potential theory on open surfaces I: analytical apparatus

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Abstract

A stable second kind integral equation formulation has been developed for the Dirichlet problem for the Laplace equation in two dimensions, with the boundary conditions specified on a collection of open curves. The performance of the obtained apparatus is illustrated with several numerical examples.

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1. Introduction

Integral equations have been one of principal tools for the numerical solution of scattering problems for more than 30 years, both in the Helmholtz and Maxwell environments. Historically, most of the equations used have been of the first kind, since numerical instabilities associated with such equations have not been critically important for the relatively small-scale problems that could be handled at the time.

The combination of improved hardware with the recent progress in the design of "fast" algorithms has changed the situation dramatically. Condition numbers of systems of linear algebraic equations resulting from the discretization of integral equations of potential theory have become critical, and the simplest way to limit such condition numbers is by starting with second kind integral equations. Hence, interest has increased in reducing scattering problems to systems of *second kind* integral equations on the boundaries of the scatterers.

During the last several years, satisfactory integral equation formulations have been constructed in both acoustic (Helmholtz equation) and electromagnetic (Maxwell's equations) environments, whenever all of

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the scattering surfaces are "closed" (i.e., scatterers have well-defined interiors, and have no infinitely thin parts). Boundary value problems for the biharmonic equation with boundary data specified on a collection of open curves have been investigated in some detail in [9–11]. However, a stable second kind integral equation formulation for scattering problems involving "open" surfaces does not appear to be present in the literature.

In this paper, we describe a stable second kind integral equation formulation for the Dirichlet problem for the Laplace equation in $\mathbb{R}^2$, with the boundary conditions specified on an "open" curve. We start with a detailed investigation of the case when the curve in question is the interval $[-1, 1]$ on the real axis; then we generalize the obtained results for the case of (reasonably) general open curves.

The layout of the paper is as follows. In Section 2, the necessary mathematical and numerical preliminaries are introduced. Section 3 contains the exact statement of the problem. Section 4 contains an informal description of the procedure. In Sections 5 and 6, we investigate the cases of the straight line segment and of the general sufficiently smooth curve, respectively. In Section 7, we describe a simple numerical implementation of the scheme described in Section 6. The performance of the algorithm is illustrated in Section 8 with several numerical examples. Finally, in Section 9 we discuss several generalizations of the approach.

2. Analytical preliminaries

In this section, we summarize several results from classical and numerical analysis to be used in the remainder of the paper. Detailed references are given in the text.

2.1. Notation

Suppose that a, b are two real numbers with $a < b$, and $f, g : [a, b] \to \mathbb{C}$ is a pair of smooth functions, and that on the interval $[a, b]$, the function g has a single simple root s. Throughout this paper, we will be repeatedly encountering expressions of the form

$$\lim_{\epsilon \to 0} \left(\int_a^{s-\epsilon} \frac{f(t)}{g(t)} \, \mathrm{d}t + \int_{s+\epsilon}^{b} \frac{f(t)}{g(t)} \, \mathrm{d}t \right), \tag{1}$$

normally referred to as principal value integrals. In a mild abuse of notation, we will refer to expressions of the form (1) simply as integrals. We will also be fairly cavalier about the spaces on which operators of the type (1) operate; whenever the properties (smoothness, boundedness, etc.) required from a function are obvious from the context, their exact specifications are omitted.

2.2. Chebyshev polynomials and Chebyshev approximation

Chebyshev polynomials are frequently encountered in numerical analysis. As is well known, Chebyshev polynomials of the first kind $T_n : [-1, 1] \to \mathbb{R} (n \geqslant 0)$ are defined by the formula

$$T_n(x) = \cos(n \arccos(x)) \tag{2}$$

and are orthogonal with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(x) \cdot g(x) \cdot \frac{1}{\sqrt{1 - x^2}} \, \mathrm{d}x. \tag{3}$$

The Chebyshev nodes x_i of degree N are the zeros of T_N defined by the formula

$$x_i = \cos\frac{(2i+1)\pi}{2N}, \quad i = 0, 1, \ldots, N-1. \tag{4}$$

Chebyshev polynomials of the second kind $U_n : [-1, 1] \to \mathbb{R}(n \geqslant 0)$ are defined by the formula

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sin(\arccos(x))} \tag{5}$$

and are orthogonal with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(x) \cdot g(x) \cdot \sqrt{1-x^2}\,\mathrm{d}x. \tag{6}$$

The Chebyshev nodes of the second kind t_j of degree N are the zeros of U_N defined by the formula

$$t_j = \cos\frac{(N-j)\pi}{N+1}, \quad j = 0, 1, \ldots, N-1. \tag{7}$$

For a sufficiently smooth function $f : [-1, 1] \to \mathbb{R}$, its Chebyshev expansion is defined by the formula

$$f(x) = \sum_{k=0}^{\infty} C_k \cdot T_k(x), \tag{8}$$

with the coefficients C_k given by the formulae

$$C_0 = \frac{1}{\pi}\int_{-1}^{1} f(x) \cdot T_0(x) \cdot (1-x^2)^{-1/2}\,\mathrm{d}x, \tag{9}$$

and

$$C_k = \frac{2}{\pi}\int_{-1}^{1} f(x) \cdot T_k(x) \cdot (1-x^2)^{-1/2}\,\mathrm{d}x, \tag{10}$$

for all $k \geqslant 1$. We will also denote by P_f^N the order $N-1$ Chebyshev approximation to the function f on the interval $[-1, 1]$, i.e., the (unique) polynomial of order $N-1$ such that $P_f^N(x_i) = f(x_i)$ for all $i = 0, 1, \ldots, N-1$, with x_i the Chebyshev nodes defined by (4).

The following lemma provides an error estimate for the Chebyshev approximation (see, for example [5]).

Lemma 1. *If $f \in C^k[-1, 1]$ (i.e., f has k continuous derivatives on the interval $[-1, 1]$), then for any $x \in [-1, 1]$,*

$$\left|P_f^N(x) - f(x)\right| = \mathrm{O}\left(\frac{1}{N^k}\right). \tag{11}$$

In particular, if f is infinitely differentiable, then the Chebyshev approximation converges superalgebraically (i.e., faster than any finite power of $1/N$ as $N \to \infty$).

2.3. The finite Hilbert transform

We will define the finite Hilbert transform $\widetilde{H}$ by the formula

$$\widetilde{H}(\varphi)(x) = \int_{-1}^{1} \frac{\varphi(t)}{t-x}\,\mathrm{d}t. \tag{12}$$

We then define the operator $\widetilde{K} : C^2[-1,1] \to L^2(-\infty,\infty)$ by the formula

$$\widetilde{K}(\varphi)(x) = \lim_{\epsilon\to 0}\left(\int_{-1}^{x-\epsilon}\frac{\varphi(t)}{(t-x)^2}\,\mathrm{d}t + \int_{x+\epsilon}^{1}\frac{\varphi(t)}{(t-x)^2}\,\mathrm{d}t - \frac{2\varphi(x)}{\epsilon}\right), \tag{13}$$

and observe that the limit (13) is often referred to as the finite part integral

$$\text{f.p.}\int_{-1}^{1}\frac{\varphi(t)}{(t-x)^2}\,\mathrm{d}t \tag{14}$$

(see, for example Hadamard [8]).

The following theorem can be found in [13]; it provides sufficient conditions for the existence of the finite part integral (14), and establishes a connection between the finite Hilbert transform (12) and the finite part integral.

Theorem 2. *For any $\varphi \in C^2[-1,1]$, the limit* (13) *is a square-integrable function of x. Furthermore*,

$$\widetilde{K}(\varphi) = D \circ \widetilde{H}(\varphi), \tag{15}$$

with $D = \frac{\mathrm{d}}{\mathrm{d}x}$ the differentiation operator.

The following theorem (see, for example [21]) describes the inverse of the operator $\widetilde{H}$, to the extent that such an inverse exists

Theorem 3. *The null space of the operator $\widetilde{H}$ is spanned by the function $1/\sqrt{1-x^2}$. Furthermore, for any function $f \in L^p[-1,1]$ with $p > 1$, all solutions of the equation*

$$\widetilde{H}(\varphi) = f \tag{16}$$

are given by the formula

$$\varphi(x) = -\frac{1}{\pi^2}T^{-1}\circ\widetilde{H}\circ T(f)(x) + \frac{C}{\sqrt{1-x^2}}, \tag{17}$$

with C an arbitrary constant, and the operator $T : L^p[-1,1] \to L^p[-1,1]$ defined by the formula

$$T(f)(x) = \sqrt{1-x^2}\cdot f(x). \tag{18}$$

Applying Theorem 3 twice, we immediately obtain the following corollary:

Corollary 4. *For any $f \in C^1[-1,1]$, all solutions of the equation*

$$\widetilde{H}\circ\widetilde{H}(\varphi) = \widetilde{H}^2(\varphi) = f \tag{19}$$

are given by the formula

$$\varphi(x) = \frac{1}{\pi^4}T^{-1}\circ\widetilde{H}^2\circ T(f)(x) + \frac{C_0}{\sqrt{1-x^2}} + \frac{C_1}{\sqrt{1-x^2}}\cdot\log\frac{1+x}{1-x}, \tag{20}$$

with C_0, C_1 two arbitrary constants.

2.4. Several elementary identities

In this section, we collect several identities from classical analysis to be used in the remainder of the paper. Lemma 5 states a well-known fact about the two-dimensional Poisson kernel $y/(x^2+y^2)$; it can be found in (for example) [19]. Lemma 6 provides explicit expressions for the finite Hilbert transform operating on Chebyshev polynomials, where (22) is a direct consequence of Lemma 3, and (23), (24) can be found in [2]. Lemma 7 lists several standard definite integrals; all can be found (in a somewhat different form) in [6]. Finally, Lemma 8 follows from the definition of curvature $c(t)$ found in elementary differential geometry (cf. [3]).

Lemma 5. *Suppose that $\sigma \in L^p[-1,1]$ $(p \geqslant 1)$. Then*

$$\lim_{y\to 0} \int_{-1}^{1} \frac{|y|}{\pi((x-t)^2+y^2)} \cdot \sigma(t)\,\mathrm{d}t = \sigma(x), \tag{21}$$

for almost all $x \in [-1,1]$.

Lemma 6. *For any $x \in (-1,1)$,*

$$\int_{-1}^{1} \frac{1}{t-x} \cdot \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t = 0, \tag{22}$$

and

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x} \cdot U_{n-1}(t)\,\mathrm{d}t = -\pi \cdot T_n(x), \tag{23}$$

$$\int_{-1}^{1} \frac{1}{t-x} \cdot \frac{1}{\sqrt{1-t^2}} \cdot T_n(t)\,\mathrm{d}t = \pi \cdot U_{n-1}(x), \tag{24}$$

for any $n \geqslant 1$.

Lemma 7.

1. *For any $x,t \in (-1,1)$ and $x \neq t$,*

$$\int_{-1}^{1} \frac{1}{(s-x)(s-t)}\,\mathrm{d}s = \frac{\log\frac{1-x}{1-t} - \log\frac{1+x}{1+t}}{x-t}. \tag{25}$$

2. *For any $(x,y) \in \mathbb{R}^2 \setminus [-1,1]$ and $t \in (-1,1)$,*

$$\int_{-1}^{1} \frac{(s-x)}{((s-x)^2+y^2)(s-t)}\,\mathrm{d}s = \frac{|y| \cdot \left(\arctan\left(\frac{1-x}{|y|}\right) + \arctan\left(\frac{1+x}{|y|}\right)\right)}{((x-t)^2+y^2)} + \frac{(x-t)\cdot\left(\log\frac{(1-x)^2+y^2}{(1-t)^2} - \log\frac{(1+x)^2+y^2}{(1+t)^2}\right)}{2((x-t)^2+y^2)}. \tag{26}$$

3. *For any $x \in (-1,1)$,*

$$\int_{-1}^{1} \log|x-t| \cdot \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t = -\pi \cdot \log 2, \tag{27}$$

$$\int_{-1}^{1} \log|x-t| \cdot \frac{t}{\sqrt{1-t^2}}\,\mathrm{d}t = -\pi \cdot x, \tag{28}$$

$$\int_{-1}^{1} \frac{1}{t-x} \cdot \frac{1}{\sqrt{1-t^2}} \cdot \log(1+t)\,\mathrm{d}t = \frac{\pi \cdot \arccos x}{\sqrt{1-x^2}}, \tag{29}$$

$$\int_{-1}^{1} \frac{1}{t-x} \cdot \frac{1}{\sqrt{1-t^2}} \cdot \log(1-t)\,\mathrm{d}t = \frac{\pi \cdot (\arccos(x)-\pi)}{\sqrt{1-x^2}}, \tag{30}$$

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x} \cdot \log(1+t)\,\mathrm{d}t = \pi \cdot (\arccos(x) \cdot \sqrt{1-x^2} + \log(2) \cdot x - 1), \tag{31}$$

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x} \cdot \log(1-t)\,\mathrm{d}t = \pi \cdot ((\arccos(x)-\pi) \cdot \sqrt{1-x^2} + \log(2) \cdot x + 1). \tag{32}$$

Lemma 8. *Suppose that $\gamma : [0, L] \to \mathbb{R}^2$ is a sufficiently smooth curve parametrized by its arc length with the unit normal and the unit tangent vectors at $\gamma(t)$ denoted by $N(t)$ and $T(t)$, respectively. Suppose further that the function $u : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable. Then at the point $\gamma(t)$, the Laplacian of u is given by the formula*

$$\Delta u = N \cdot \nabla\nabla u \cdot N - c(t) N \cdot \nabla u + \frac{\mathrm{d}^2}{\mathrm{d}t^2} u(\gamma(t)) = \frac{\partial^2 u}{\partial N(t)^2} - c(t) \cdot \frac{\partial u}{\partial N(t)} + \frac{\partial^2 u}{\partial T(t)^2}, \tag{33}$$

where the curvature $c(t)$ at $\gamma(t)$ is defined by $\mathrm{d}^2\gamma/\mathrm{d}t^2 = c(t)N(t)$.

2.5. The Poincaré–Bertrand formula

For a fixed point $x \in (-1, 1)$, we will consider two repeated integrals

$$A = \int_{-1}^{1} \frac{\varphi_1(t)}{t-x} \cdot \left(\int_{-1}^{1} \frac{\varphi_2(s)}{s-t}\,\mathrm{d}s \right) \mathrm{d}t, \tag{34}$$

$$B = \int_{-1}^{1} \varphi_2(s) \cdot \left(\int_{-1}^{1} \frac{\varphi_1(t)}{(t-x)(s-t)}\,\mathrm{d}t \right) \mathrm{d}s, \tag{35}$$

differing from each other only in the order of integration. Both integrals exist almost everywhere for a fairly broad class of functions. However, they are not, in general, equal to one another. The following lemma establishes the connection between them (see, for example [17,21]; the result is usually referred to as the Poincaré–Bertrand formula.

Lemma 9. *Suppose that $\varphi_1 \in L^p[-1, 1]$, $\varphi_2 \in L^q[-1, 1]$. Then if*

$$\frac{1}{p} + \frac{1}{q} < 1, \tag{36}$$

then

$$\int_{-1}^{1} \frac{\varphi_1(t)}{t-x} \cdot \left(\int_{-1}^{1} \frac{\varphi_2(s)}{s-t} \, \mathrm{d}s \right) \mathrm{d}t = -\pi^2 \cdot \varphi_1(x) \cdot \varphi_2(x) + \int_{-1}^{1} \varphi_2(s) \cdot \left(\int_{-1}^{1} \frac{\varphi_1(t)}{(t-x)(s-t)} \, \mathrm{d}t \right) \mathrm{d}s \tag{37}$$

for almost all $x \in (-1, 1)$.

2.6. Potential theory

In this section, we introduce some terminology standard in potential theory and state several technical lemmas to be used subsequently. We will define the potential $G_{x_0} : \mathbb{R}^2 \setminus \{x_0\} \to \mathbb{R}$ of a unit charge located at the point $x_0 \in \mathbb{R}^2$ by the formula

$$G_{x_0}(x) = \log(\|x - x_0\|). \tag{38}$$

Suppose that $\gamma : [0, L] \to \mathbb{R}^2$ is a sufficiently smooth curve parametrized by its arc length, and that γ is an open curve (i.e., $\gamma(0) \neq \gamma(L)$). The image of γ will be denoted by Γ, and the unit normal and the unit tangent vectors to γ at the point $\gamma(t)$ will be denoted by $N(t)$ and $T(t)$, respectively. Given an integrable function $\sigma : [0, L] \to \mathbb{R}$, we will refer to the functions $S_{\gamma,\sigma} : \mathbb{R}^2 \to \mathbb{R}$ and $D_{\gamma,\sigma}, Q_{\gamma,\sigma} : \mathbb{R}^2 \setminus \Gamma \to \mathbb{R}$, defined by the formulae

$$S_{\gamma,\sigma}(x) = \int_0^L G_{\gamma(t)}(x) \cdot \sigma(t) \, \mathrm{d}t, \tag{39}$$

$$D_{\gamma,\sigma}(x) = \int_0^L \frac{\partial G_{\gamma(t)}(x)}{\partial N(t)} \cdot \sigma(t) \, \mathrm{d}t, \tag{40}$$

$$Q_{\gamma,\sigma}(x) = \int_0^L \frac{\partial^2 G_{\gamma(t)}(x)}{\partial N(t)^2} \cdot \sigma(t) \, \mathrm{d}t, \tag{41}$$

as the single, double, and quadruple layer potentials, respectively.

The functions $(\partial G_{\gamma(t)}(x))/(\partial N(t)), (\partial^2 G_{\gamma(t)}(x))/(\partial N(t)^2) : \mathbb{R}^2 \setminus \gamma(t) \to \mathbb{R}$ are often referred to as the dipole and quadrupole potentials, respectively. Obviously,

$$\frac{\partial G_{\gamma(t)}(x)}{\partial N(t)} = -\frac{\langle N(t), x - \gamma(t) \rangle}{\|x - \gamma(t)\|^2}, \tag{42}$$

$$\frac{\partial^2 G_{\gamma(t)}(x)}{\partial N(t)^2} = -\frac{2\langle N(t), x - \gamma(t) \rangle^2}{\|x - \gamma(t)\|^4} + \frac{1}{\|x - \gamma(t)\|^2}. \tag{43}$$

In particular, if γ is a straight line segment $I_L = [0, L]$ on the real axis, then

$$\frac{\partial G_{I(s+t)}(I(s) - h \cdot N(s))}{\partial N(s+t)} = \frac{h}{h^2 + t^2}, \tag{44}$$

$$\frac{\partial^2 G_{I(s+t)}(I(s) - h \cdot N(s))}{\partial N(s+t)^2} = \frac{t^2 - h^2}{(h^2 + t^2)^2}. \tag{45}$$

The following two lemmas can be found in [14]. Lemma 10 states a standard fact from elementary differential geometry of curves; Lemma 11 describes the local behavior on a curve of the potential of a quadrupole located on that curve and oriented normally to it.

Lemma 10. *Suppose that $\gamma : [0, L] \to \mathbb{R}^2$ is a sufficiently smooth curve parametrized by its arc length with the unit normal and the unit tangent vectors at $\gamma(t)$ denoted by $N(t)$ and $T(t)$, respectively. Then, there exist a positive real number β (dependent on γ), and two continuously differentiable functions $f, g : (-\beta, \beta) \to \mathbb{R}$ (dependent on γ), such that for any $t \in [0, L]$,*

$$\gamma(t+s) - \gamma(t) = \left(s - \frac{c(t)^2 \cdot s^3}{6} + s^4 \cdot f(s)\right) \cdot T(t) + \left(\frac{c(t) \cdot s^2}{2} + s^3 \cdot g(s)\right) \cdot N(t), \tag{46}$$

for all $s \in (-\beta, \beta)$, where $c(t)$ in (46) *is the curvature of γ at the point $\gamma(t)$.*

Lemma 11. *Suppose that $\gamma : [0, L] \to \mathbb{R}^2$ is a sufficiently smooth curve parametrized by its arc length. Then, there exist real positive numbers A, β, h_0 such that for any $s \in [0, L]$,*

$$\left| \frac{\partial^2 G_{\gamma(s+t)}(\gamma(s) - h \cdot N(s))}{\partial N(s+t)^2} - \frac{t^2 - h^2}{(h^2 + t^2)^2} - \frac{c \cdot h \cdot t^2 \cdot (5h^2 + t^2)}{(h^2 + t^2)^3} \right| \leqslant A, \tag{47}$$

for all $t \in (-\beta, \beta)$, $0 \leqslant h < h_0$, where the coefficient c in (47) *is the positive curvature of γ at the point $\gamma(s)$.*

Similarly, the following lemma describes the local behavior on a curve of the potential of a dipole located on that curve and oriented normally to it; it also describes the local behavior on a curve of the tangential derivative of the potential of a charge located on that curve. Its proof is virtually identical to that of Lemma 11.

Lemma 12. *Under the conditions of Lemma* 11, *there exist real positive numbers A, β, h_0 such that for any $s \in [0, L]$,*

$$\left| \frac{\partial G_{\gamma(s+t)}(\gamma(s) - h \cdot N(s))}{\partial N(s+t)} - \frac{h}{h^2 + t^2} \right| \leqslant A, \tag{48}$$

$$\left| \frac{\partial G_{\gamma(s+t)}(\gamma(s) - h \cdot N(s))}{\partial T(s+t)} - \frac{t}{h^2 + t^2} \right| \leqslant A, \tag{49}$$

for all $t \in (-\beta, \beta)$, $0 \leqslant h < h_0$.

We will define the function $M_{\gamma,\sigma} : \mathbb{R}^2 \setminus \Gamma \to \mathbb{R}$ by the formula

$$M_{\gamma,\sigma}(x) = Q_{\gamma,\sigma}(x) - D_{\gamma,c\sigma}(x) = \int_0^L \left(\frac{\partial^2 G_{\gamma(t)}(x)}{\partial N(t)^2} - c(t) \cdot \frac{\partial G_{\gamma(t)}(x)}{\partial N(t)} \right) \cdot \sigma(t)\,\mathrm{d}t, \tag{50}$$

for all $x \in \mathbb{R}^2 \setminus \Gamma$ and observe that $M_{\gamma,\sigma}$ is the difference of a quadruple layer potential and a weighted double layer potential with the weight equal to the curvature $c(t)$. The following theorem is a direct consequence of Lemmas 11 and 12; it states that under certain conditions the function $M_{\gamma,\sigma}$ defined by (50) can be continuously extended to the whole plane $\mathbb{R}^2$.

Theorem 13. *Suppose that $\gamma : [0, L] \to \mathbb{R}^2$ is a sufficiently smooth open curve parametrized by its arc length, and that $\sigma : [0, L] \to \mathbb{R}$ is a function continuous on $[0, L]$, whose second derivative is continuous on $(0, L)$. Then the function $M_{\gamma,\sigma}$ can be continuously extended to $\mathbb{R}^2 \setminus \{\gamma(0), \gamma(L)\}$ with the limiting value on $\gamma(0, L)$ defined by the formula*

$$M_{\gamma,\sigma}(\gamma(x)) = \text{f.p.} \int_0^L \frac{\partial^2 G_{\gamma(t)}(\gamma(x))}{\partial N(t)^2} \cdot \sigma(t)\,\mathrm{d}t - \int_0^L c(t) \cdot \frac{\partial G_{\gamma(t)}(\gamma(x))}{\partial N(t)} \cdot \sigma(t)\,\mathrm{d}t \tag{51}$$

for all $x \in (0, L)$. Furthermore, if σ satisfies the additional condition that

$$|\sigma(x)| \leqslant C \cdot (x \cdot (L - x))^{\alpha}, \tag{52}$$

with some $C > 0$, $\alpha > 1$ for all $x \in [0, L]$, then $M_{\gamma,\sigma}$ can be further continuously extended to $\mathbb{R}^2$ with the limiting values on $\gamma(0)$, $\gamma(L)$ given by the improper integrals

$$M_{\gamma,\sigma}(\gamma(0)) = \int_0^L \left(\frac{\partial^2 G_{\gamma(t)}(\gamma(0))}{\partial N(t)^2} - c(t) \cdot \frac{\partial G_{\gamma(t)}(\gamma(0))}{\partial N(t)} \right) \cdot \sigma(t)\,\mathrm{d}t, \tag{53}$$

$$M_{\gamma,\sigma}(\gamma(L)) = \int_0^L \left(\frac{\partial^2 G_{\gamma(t)}(\gamma(L))}{\partial N(t)^2} - c(t) \cdot \frac{\partial G_{\gamma(t)}(\gamma(L))}{\partial N(t)} \right) \cdot \sigma(t)\,\mathrm{d}t, \tag{54}$$

respectively.

Definition 14. We will denote by E the linear subspace of $C[0, L]$, consisting of functions σ satisfying the following two conditions:
1. σ is twice continuously differentiable on $(0, L)$;
2. σ satisfies the condition (52).

We then define the integral operator $M_\gamma : E \to C[0, L]$ via the formula

$$M_\gamma(\sigma)(x) = M_{\gamma,\sigma}(\gamma(x)). \tag{55}$$

The following lemma states that the operator M_γ on a sufficiently smooth open curve γ is a compact perturbation of the same operator M_{I_L} on the line segment $I_L = [0, L]$.

Lemma 15. *Suppose that $\gamma : [0, L] \to \mathbb{R}^2$ is a sufficiently smooth open curve parametrized by its arc length. Suppose further that the operator $R_\gamma : C[0, L] \to C[0, L]$ is defined by the formula*

$$R_\gamma(\sigma)(x) = \int_0^L r(x, t) \cdot \sigma(t)\,\mathrm{d}t \tag{56}$$

with the function $r : [0, L] \times [0, L] \to \mathbb{R}$ defined by the formula

$$r(x, t) = \frac{\partial^2 G_{\gamma(t)}(\gamma(x))}{\partial N(t)^2} - c(t) \cdot \frac{\partial G_{\gamma(t)}(\gamma(x))}{\partial N(t)} - \frac{\partial^2 G_{I_L(t)}(x)}{\partial N(t)^2} \tag{57}$$

for all $x \neq t$, and by the formula

$$r(t, t) = \frac{c(t)^2}{12} \tag{58}$$

for all $x = t$, with $c(t)$ denoting the curvature of γ at the point $\gamma(t)$. Then

$$r(x, t) = -\frac{2\langle N(t), \gamma(x) - \gamma(t)\rangle^2}{\|\gamma(x) - \gamma(t)\|^4} + \frac{1}{\|\gamma(x) - \gamma(t)\|^2} + c(t) \cdot \frac{\langle N(t), \gamma(x) - \gamma(t)\rangle}{\|\gamma(x) - \gamma(t)\|^2} - \frac{1}{(x - t)^2} \tag{59}$$

for all $x \neq t$. Furthermore, r is continuous on $[0, L] \times [0, L]$, so that the operator R_γ is compact. Finally, if $\sigma \in E$ (see Definition 14 above), then

$$M_\gamma(\sigma)(x) = M_{I_L}(\sigma)(x) + R_\gamma(\sigma)(x). \tag{60}$$

Proof. Eq. (60) follows directly from the combination of (51), (56), (57) and the fact that the curvature is zero everywhere on the line segment I_L. (59) is a direct consequence of (42), (43), (45), (57). In order to prove that r is continuous on $[0,L]\times[0,L]$, we start with observing that since $\gamma\in C^2[0,L]$, it is sufficient to demonstrate that

$$\lim_{s\to 0} r(t+s,t) = \frac{c(t)^2}{12}. \tag{61}$$

Replacing x in (59) with $t+s$, we obtain

$$r(t+s,t) = -\frac{2\langle N(t), \gamma(t+s)-\gamma(t)\rangle^2}{\|\gamma(t+s)-\gamma(t)\|^4} + \frac{1}{\|\gamma(t+s)-\gamma(t)\|^2} + c(t)\cdot\frac{\langle N(t),\gamma(t+s)-\gamma(t)\rangle}{\|\gamma(t+s)-\gamma(t)\|^2} - \frac{1}{s^2}. \tag{62}$$

Substituting (46) into (62), we have

$$r(t+s,t) = -\frac{2p(s)^2}{d(s)^2} + c(t)\cdot\frac{p(s)}{d(s)} + \frac{1-d(s)}{s^2\cdot d(s)}, \tag{63}$$

where the functions $p,d:(-\beta,\beta)\to\mathbb{R}$ are given be the formulae

$$p(s) = \frac{c(t)}{2} + s\cdot g(s), \tag{64}$$

$$d(s) = \left(1 - \frac{c(t)^2\cdot s^2}{6} + s^3\cdot f(s)\right)^2 + \left(\frac{c(t)\cdot s}{2} + s^2\cdot g(s)\right)^2, \tag{65}$$

with β a positive real number, and the functions f, g provided by Lemma 10. Since f, g are continuously differentiable on $(-\beta,\beta)$ (see Lemma 10), we have

$$\lim_{s\to 0}\frac{p(s)}{d(s)} = \frac{c(t)}{2}, \tag{66}$$

$$\lim_{s\to 0}\frac{1-d(s)}{s^2\cdot d(s)} = \frac{c(t)^2}{12}. \tag{67}$$

Now, we obtain (61) by substituting (66), (67) into (63). □

Remark 16. A somewhat involved analysis shows that for any $k\geqslant 1$ and $\gamma\in C^{k+2}[0,L]$, the function r (see (57) above) is k times continuously differentiable. The proof of this fact is technical, and the fact itself is peripheral to the purpose of this paper; thus, the proof is omitted.

3. The exact statement of the problem

Suppose that γ is a sufficiently smooth open curve, and that the image of γ is denoted by Γ. We will denote by S_γ the set of continuous functions on $\mathbb{R}^2$ with continuous second derivatives in the complement of Γ, i.e.,

$$S_\gamma = C^2(\mathbb{R}^2\setminus\Gamma)\cap C(\mathbb{R}^2). \tag{68}$$

We will consider the Dirichlet problem for the Laplace equation in $\mathbb{R}^2$, with the boundary conditions specified on γ:

Given a function $f : \Gamma \to \mathbb{R}$, find a bounded solution $u \in S_\gamma$ to the Laplace equation

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma \tag{69}$$

satisfying the Dirichlet boundary condition

$$u = f \quad \text{on } \Gamma. \tag{70}$$

The following theorem can be found in [15].

Theorem 17. *If $f \in C^2(\Gamma)$, then there exists a unique bounded solution in S_γ to the problem* (69) *and* (70).

Remark 18. Certain physical problems lead to modifications of the problem (69) and (70). For example, the boundedness of the solution at infinity might be replaced with logarithmic growth, the boundary might consist of several disjoint components, etc. Extensions of Theorem 17 to these cases are straightforward, and can be found, for example, in [17].

4. Analytical apparatus I: informal description

In this section, we will present an informal description of the procedure. We assume that $\gamma : [-1, 1] \to \mathbb{R}^2$ is a sufficiently smooth "open" (i.e., $\gamma(-1) \neq \gamma(1)$) curve with the parametrization

$$\gamma(t) = \widetilde{\gamma}\left(\frac{L}{2} \cdot (t+1)\right), \tag{71}$$

where L is the total arc length of the curve, and $\widetilde{\gamma} : [0, L] \to \mathbb{R}^2$ is the same curve parametrized by its arc length. The image of γ will be denoted by Γ. We start with observing that the solution u of the Dirichlet problem (69) and (70) must satisfy the following four conditions:

(a) u is harmonic in $\mathbb{R}^2 \setminus \Gamma$;
(b) u is bounded at infinity;
(c) u is continuous across Γ;
(d) u is equal to the prescribed data f on Γ.

Our goal is to construct a second kind integral formulation for the Dirichlet problem (69) and (70). Standard approaches in classical potential theory call for representing u in $\mathbb{R}^2 \setminus \Gamma$ via single or double layer potentials so that conditions (a), (b) are automatically satisfied, and conditions (c), (d) lead to a boundary integral equation via the so-called jump relations of single and double layer potentials (see, for example [16]). However, in the case of an open curve, if u is represented via a double layer potential, the condition (c) cannot be satisfied since any non-zero double layer potential has a jump across the boundary; and if u is represented via a single layer potential, while the single layer potential can be continuously extended across the boundary, the condition (d) will lead to an integral equation of the *first* kind. Hence, classical tools of potential theory turn out to be insufficient for dealing with open surface problems.

It is shown in [14] that the quadruple layer potential has a jump across the boundary which is proportional to the curvature of the curve. Combining this observation with the well-known fact that the double layer potential has a jump across the boundary which is independent of the curvature, we observe that the sum of a quadruple layer potential and a weighted double layer potential with the weight equal to the curvature given by the formula

$$\int_{-1}^{1}\left(\frac{\partial^2 G_{\gamma(t)}(x)}{\partial N(t)^2} - c(t)\cdot\frac{\partial G_{\gamma(t)}(x)}{\partial N(t)}\right)\cdot\sigma(t)\,\mathrm{d}t \tag{72}$$

can be continuously extended across the boundary. However, if u is represented via (72), then the condition (d) will lead to a *hypersingular* integral equation. It is also shown in [14] that the product of the hypersingular integral operator with the single layer potential operator is a second kind integral operator in the case of a closed boundary. Thus, one is naturally lead to consider the operator P_γ defined by the formula

$$P_\gamma(\sigma)(x) = \int_{-1}^{1}\left(\frac{\partial^2 G_{\gamma(t)}(x)}{\partial N(t)^2} - c(t)\cdot\frac{\partial G_{\gamma(t)}(x)}{\partial N(t)}\right)\cdot\left(\int_{-1}^{1}\log|t-s|\cdot\sigma(s)\,\mathrm{d}s\right)\mathrm{d}t. \tag{73}$$

Obviously, $P_\gamma(\sigma)$ is not defined when $x \in \Gamma$, and we will define the operator B_γ by the formula

$$B_\gamma(\sigma)(t) = \lim_{x\to\gamma(t)} P_\gamma(\sigma)(x). \tag{74}$$

In the special case when γ is the interval $I = [-1, 1]$ on the real axis, (73) assumes the form

$$P_I(\sigma)(x,y) = \frac{1}{2}\int_{-1}^{1}\frac{\partial^2}{\partial y^2}\log((x-s)^2+y^2)\cdot\left(\int_{-1}^{1}\log|s-t|\cdot\sigma(t)\,\mathrm{d}t\right)\mathrm{d}s, \tag{75}$$

and the operator B_I is defined by the formula

$$B_I(\sigma)(x) = \lim_{y\to 0} P_I(\sigma)(x,y). \tag{76}$$

The operator B_I turns out to have a remarkably simple analytical structure (see Section 5.4 below); its natural domain consists of functions of the form

$$\frac{1}{\sqrt{1-x^2}}\cdot\varphi(x) + \frac{1}{\sqrt{1-x^2}}\cdot\log\frac{1+x}{1-x}\cdot\psi(x), \tag{77}$$

with φ, ψ smooth functions, and when restricted to functions of the form (77), it has a null-space of dimension 2, spanned by the functions

$$\frac{1}{\sqrt{1-x^2}}, \tag{78}$$

$$\frac{1}{\sqrt{1-x^2}}\cdot\log\frac{1+x}{1-x}. \tag{79}$$

In Section 5.4, we construct a generalized (in the appropriate sense) inverse of B_I; in a mild abuse of notation, we will refer to it as B_I^{-1}.

Now, if we represent the solution of the Problem (69) and (70) in the form

$$u(x) = P_\gamma(\sigma)(x), \tag{80}$$

then the conditions (c) and (d) will lead to the equation

$$B_\gamma(\sigma)(t) = f(t), \tag{81}$$

with σ the unknown density. It turns out that (81) behaves *almost* like an integral equation of the second kind; the only problem is that the kernel of B_γ is strongly singular at the ends. Fortunately, the operator

$$\widetilde{B}_\gamma = B_\gamma \circ B_I^{-1}, \tag{82}$$

restricted to smooth functions, *is* a sum of the identity and a compact operator. In other words, $\widetilde{B}_\gamma$ is a *second kind* integral operator. Therefore, our representation for the solution of the Problem (69) and (70) takes the form

$$u(x) = \widetilde{P}_\gamma(\eta)(x) = P_\gamma \circ B_I^{-1}(\eta)(x), \tag{83}$$

with η the solution of the integral equation

$$\widetilde{B}_\gamma(\eta)(t) = f(t). \tag{84}$$

Finally, we remark that minor complications arise from the non-uniqueness of B_I^{-1} (see (78) and (79) above); they are resolved in Section 6.3.

5. Analytical apparatus II: open surface problem for the line segment $I = [-1,1]$

5.1. The integral operator P_I

Definition 19. We will denote by F_I the set of functions $\sigma : (-1,1) \to \mathbb{R}$ of the form

$$\sigma(x) = \frac{1}{\sqrt{1-x^2}} \cdot \varphi(x) + \frac{1}{\sqrt{1-x^2}} \cdot \log\frac{1+x}{1-x} \cdot \psi(x), \tag{85}$$

with $\varphi, \psi : [-1,1] \to \mathbb{R}$ twice continuously differentiable, and satisfying the conditions

$$\int_{-1}^{1} \log|1+t| \cdot \sigma(t)\,\mathrm{d}t = 0, \tag{86}$$

$$\int_{-1}^{1} \log|1-t| \cdot \sigma(t)\,\mathrm{d}t = 0. \tag{87}$$

We will consider the integral operator $P_I : F_I \to C^2(\mathbb{R}^2 \setminus I)$ defined by the formula

$$P_I(\sigma)(x,y) = \int_{-1}^{1} K_I(x,y,t) \cdot \sigma(t)\,\mathrm{d}t = \frac{1}{2}\int_{-1}^{1} \frac{\partial^2}{\partial y^2} \log((x-s)^2+y^2) \cdot \left(\int_{-1}^{1} \log|s-t| \cdot \sigma(t)\,\mathrm{d}t\right)\mathrm{d}s. \tag{88}$$

Obviously, P_I converts a function $\sigma \in F_I$ into a quadruple layer potential whose density $D(\sigma)$ is in turn represented by a single layer potential

$$D(\sigma)(x) = \int_{-1}^{1} \log|x-t| \cdot \sigma(t)\,\mathrm{d}t. \tag{89}$$

The following lemma provides an explicit expression for the kernel K_I of P_I.

Lemma 20. *For any* $\sigma \in F_I$,

$$K_I(x,y,t) = \frac{|y| \cdot \left(\arctan\left(\frac{1-x}{|y|}\right) + \arctan\left(\frac{1+x}{|y}\right)\right)}{((x-t)^2+y^2)} + \frac{(x-t) \cdot \left(\log\frac{(1-x)^2+y^2}{(1-t)^2} - \log\frac{(1+x)^2+y^2}{(1+t)^2}\right)}{2((x-t)^2+y^2)}, \tag{90}$$

for any $(x,y) \in \mathbb{R}^2 \setminus I$ *and any* $t \in (-1,1)$.

Proof. Since $\log((x-s)^2+y^2)$ satisfies the Laplace equation for any $(x,y) \neq (s,0)$, we have

$$\frac{\partial^2}{\partial y^2}\log((x-s)^2+y^2) = -\frac{\partial^2}{\partial s^2}\log((x-s)^2+y^2); \tag{91}$$

substituting (91) into (88) and integrating by parts once, we obtain

$$\begin{aligned} P_I(\sigma)(x,y) = {} & \frac{1}{2}\int_{-1}^{1}\frac{\partial}{\partial s}\log((x-s)^2+y^2)\cdot\left(\int_{-1}^{1}\frac{\partial}{\partial s}\log|s-t|\cdot\sigma(t)\,\mathrm{d}t\right)\mathrm{d}s \\ & -\frac{(1-x)}{(x-1)^2+y^2}\cdot\int_{-1}^{1}\log|1-t|\cdot\sigma(t)\,\mathrm{d}t - \frac{(1+x)}{(x+1)^2+y^2}\cdot\int_{-1}^{1}\log|1+t|\cdot\sigma(t)\,\mathrm{d}t. \end{aligned} \tag{92}$$

Combining (92) with (86), (87) and changing the order of integration, we have

$$P_I(\sigma)(x,y) = \int_{-1}^{1}\left(\frac{1}{2}\int_{-1}^{1}\frac{\partial}{\partial s}\log((s-x)^2+y^2)\cdot\frac{\partial}{\partial s}\log|s-t|\,\mathrm{d}s\right)\cdot\sigma(t)\,\mathrm{d}t. \tag{93}$$

Hence,

$$K_I(x,y,t) = \frac{1}{2}\int_{-1}^{1}\frac{\partial}{\partial s}\log((s-x)^2+y^2)\cdot\frac{\partial}{\partial s}\log|s-t|\,\mathrm{d}s = \int_{-1}^{1}\frac{(s-x)}{((s-x)^2+y^2)(s-t)}\,\mathrm{d}s. \tag{94}$$

Now, (90) follows immediately from the combination of (26) and (94). □

5.2. The boundary integral operator B_I

We will define the integral operator $B_I : F_I \to L^1[-1,1]$ (see (85)) by the formula

$$B_I(\sigma)(x) = \lim_{y\to 0} P_I(\sigma)(x,y) = \lim_{y\to 0}\int_{-1}^{1} K_I(x,y,t)\cdot\sigma(t)\,\mathrm{d}t. \tag{95}$$

The following lemma provides an explicit expression for B_I.

Lemma 21. *For any $x \in (-1,1)$,*

$$B_I(\sigma)(x) = \pi^2\cdot\sigma(x) + \int_{-1}^{1}\frac{\log\frac{1-x}{1-t} - \log\frac{1+x}{1+t}}{x-t}\cdot\sigma(t)\,\mathrm{d}t. \tag{96}$$

Proof. Substituting (90) into (95), we obtain

$$\begin{aligned} B_I(\sigma)(x) = {} & \lim_{y\to 0}\int_{-1}^{1}\frac{|y|\cdot\left(\arctan\left(\frac{1-x}{|y|}\right)+\arctan\left(\frac{1+x}{|y|}\right)\right)}{((x-t)^2+y^2)}\cdot\sigma(t)\,\mathrm{d}t \\ & + \lim_{y\to 0}\int_{-1}^{1}\frac{(x-t)\cdot\left(\log\frac{(1-x)^2+y^2}{(1-t)^2} - \log\frac{(1+x)^2+y^2}{(1+t)^2}\right)}{2((x-t)^2+y^2)}\cdot\sigma(t)\,\mathrm{d}t. \end{aligned} \tag{97}$$

Combining (21) with the trivial identity

$$\lim_{y\to 0} \arctan\left(\frac{1-x}{|y|}\right) + \arctan\left(\frac{1+x}{|y|}\right) = \pi, \quad x \in (-1,1), \tag{98}$$

we have

$$\lim_{y\to 0} \int_{-1}^{1} \frac{|y| \cdot \left(\arctan\left(\frac{1-x}{|y|}\right) + \arctan\left(\frac{1+x}{|y|}\right)\right)}{((x-t)^2 + y^2)} \cdot \sigma(t)\,\mathrm{d}t = \pi^2 \cdot \sigma(x). \tag{99}$$

Now, applying Lebesgue's dominated convergence theorem (see, for example [18]) to the second part of the right-hand side of (97), we have

$$\begin{aligned} &\lim_{y\to 0} \int_{-1}^{1} \frac{(x-t)\cdot\left(\log\frac{(1-x)^2+y^2}{(1-t)^2} - \log\frac{(1+x)^2+y^2}{(1+t)^2}\right)}{2((x-t)^2+y^2)} \cdot \sigma(t)\,\mathrm{d}t \\ &= \int_{-1}^{1} \lim_{y\to 0} \frac{(x-t)\cdot\left(\log\frac{(1-x)^2+y^2}{(1-t)^2} - \log\frac{(1+x)^2+y^2}{(1+t)^2}\right)}{2((x-t)^2+y^2)} \cdot \sigma(t)\,\mathrm{d}t = \int_{-1}^{1} \frac{\log\frac{1-x}{1-t} - \log\frac{1+x}{1+t}}{x-t} \cdot \sigma(t)\,\mathrm{d}t. \end{aligned} \tag{100}$$

Finally, combining (99), (100) with (97), we obtain (96). □

Remark 22. Elementary analysis shows that

$$\lim_{t\to x} \frac{\log\frac{1-x}{1-t} - \log\frac{1+x}{1+t}}{x-t} = -\frac{1}{1-x} - \frac{1}{1+x} = -\frac{2}{1-x^2}. \tag{101}$$

That is, the only singularities of the integral kernel in (96) are at the end points ± 1.

5.3. Connection between the operator B_I and the finite Hilbert transform

Lemma 23. *For any $\sigma \in F_I$ (see Definition 19),*

$$B_I(\sigma)(x) = -\widetilde{H}^2(\sigma)(x) \tag{102}$$

for all $x \in (-1,1)$.

Proof. Due to (12),

$$\widetilde{H}^2(\sigma)(x) = \int_{-1}^{1} \frac{1}{s-x} \cdot \left(\int_{-1}^{1} \frac{1}{t-s} \cdot \sigma(t)\,\mathrm{d}t\right) \mathrm{d}s. \tag{103}$$

Combining (37) with (103), we have

$$\widetilde{H}^2(\sigma)(x) = -\left(\pi^2 \cdot \sigma(x) + \int_{-1}^{1} \left(\int_{-1}^{1} \frac{1}{(s-x)(s-t)}\,\mathrm{d}s\right) \cdot \sigma(t)\,\mathrm{d}t\right). \tag{104}$$

Now, (102) follows immediately from the combination of (25), (96), (104). □

5.4. The inverse of $\widetilde{H}^2$ for Chebyshev polynomials

In Section 5.5, we will need the ability to solve equations of the form (19). However, due to Corollary 4, the solution to (19) is not unique. The purpose of this section is Theorem 28, stating that the solution to (19) *is* unique if restricted to the function space F_I (see Definition 19), and constructing such a solution.

The following lemma is a direct consequence of Corollary 4 and Lemma 6.

Lemma 24. *For any integer $n \geqslant 0$ and $x \in (-1,1)$, all solutions of the equation*

$$\widetilde{H}^2(\sigma_n) = T_n \tag{105}$$

are given by the formula

$$\sigma_n(x) = \widetilde{\sigma}_n(x) + \frac{C_0}{\sqrt{1-x^2}} + \frac{C_1}{\sqrt{1-x^2}} \cdot \log\frac{1+x}{1-x}, \tag{106}$$

with C_0, C_1 arbitrary constants, and the functions $\widetilde{\sigma}_n$ defined by the formulae:

$$\widetilde{\sigma}_0(x) = \frac{1}{\pi^3} \cdot \frac{x}{\sqrt{1-x^2}} \cdot \log\frac{1+x}{1-x}, \tag{107}$$

and

$$\widetilde{\sigma}_{2k}(x) = \frac{1}{\pi^3} \cdot \sqrt{1-x^2} \cdot \int_{-1}^{1} \frac{U_{2k-1}(t)}{t-x}\,\mathrm{d}t, \tag{108}$$

$$\widetilde{\sigma}_{2k-1}(x) = \frac{1}{\pi^3} \cdot \sqrt{1-x^2} \cdot \int_{-1}^{1} \frac{U_{2k-2}(t)}{t-x}\,\mathrm{d}t - \frac{2}{(2k-1)\pi^3} \cdot \frac{x}{\sqrt{1-x^2}}, \tag{109}$$

for all $k \geqslant 1$.

We will define the operators $J, L : C^1[-1,1] \to C[-1,1]$ via the formulae:

$$J(\varphi)(x) = \int_{-1}^{1} \log|x-t| \cdot \frac{1}{\sqrt{1-t^2}} \cdot \left(\int_{-1}^{1} \frac{\varphi(s)}{t-s}\,\mathrm{d}s \right) \mathrm{d}t, \tag{110}$$

$$L(\varphi)(x) = \int_{-1}^{1} \log|x-t| \cdot \sqrt{1-t^2} \cdot \left(\int_{-1}^{1} \frac{\varphi(s)}{t-s}\,\mathrm{d}s \right) \mathrm{d}t. \tag{111}$$

The following lemma provides explicit expressions for the derivatives of $J(\varphi)$, $L(\varphi)$, and for the values of $J(\varphi)$, $L(\varphi)$ at the points -1, 1.

Lemma 25. *For any $\varphi \in C^1[-1,1]$,*

$$J'(\varphi)(x) = -\pi^2 \cdot \frac{\varphi(x)}{\sqrt{1-x^2}}, \tag{112}$$

$$L'(\varphi)(x) = -\pi^2 \cdot \varphi(x) \cdot \sqrt{1-x^2} + \pi \cdot \int_{-1}^{1} \varphi(s)\,\mathrm{d}s, \tag{113}$$

for any $x \in (-1,1)$, and

$$J(\varphi)(-1) = \pi \cdot \int_{-1}^{1} \frac{\arccos(s)}{\sqrt{1-s^2}} \cdot \varphi(s)\,\mathrm{d}s, \tag{114}$$

$$J(\varphi)(1) = \pi \cdot \int_{-1}^{1} \frac{\arccos(s) - \pi}{\sqrt{1-s^2}} \cdot \varphi(s)\,\mathrm{d}s, \tag{115}$$

$$L(\varphi)(-1) = \pi \cdot \int_{-1}^{1} \varphi(x) \cdot (\arccos(x) \cdot \sqrt{1-x^2} + \log(2) \cdot x - 1)\,\mathrm{d}x. \tag{116}$$

$$L(\varphi)(1) = \pi \cdot \int_{-1}^{1} \varphi(x) \cdot ((\arccos(x) - \pi) \cdot \sqrt{1-x^2} + \log(2) \cdot x + 1)\,\mathrm{d}x. \tag{117}$$

Proof. The identities (114)–(117) are a direct consequence of (29)–(32) in Lemma 7, respectively. In order to prove (112), substituting (110) into $J'(\varphi)$ and interchanging the order of the differentiation and integration, we obtain

$$J'(\varphi)(x) = \int_{-1}^{1} \frac{1}{x-t} \cdot \frac{1}{\sqrt{1-t^2}} \cdot \left(\int_{-1}^{1} \frac{\varphi(s)}{t-s}\,\mathrm{d}s \right) \mathrm{d}t. \tag{118}$$

Applying (37) to the right-hand side of (118), we have

$$\begin{aligned} J'(\varphi)(x) = & -\pi^2 \cdot \frac{\varphi(x)}{\sqrt{1-x^2}} + \int_{-1}^{1} \frac{\varphi(s)}{x-s} \cdot \left(\int_{-1}^{1} \frac{1}{t-s} \cdot \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t \right) \mathrm{d}s \\ & - \int_{-1}^{1} \frac{\varphi(s)}{x-s} \cdot \left(\int_{-1}^{1} \frac{1}{t-x} \cdot \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t \right) \mathrm{d}s. \end{aligned} \tag{119}$$

Now, (112) follows immediately from the combination of (22), (119). The proof of (113) is virtually identical to that of (112). □

The following lemma provides explicit expressions for $J(T_n)$, with $n = 0, 1, 2, \ldots$

Lemma 26. *For any* $x \in [-1, 1]$,

$$J(T_0)(x) = -\frac{\pi^3}{2} + \pi^2 \cdot \arccos(x) \tag{120}$$

and

$$J(T_{2n})(x) = \frac{\pi^2}{2n} \cdot \sqrt{1-x^2} \cdot U_{2n-1}(x), \tag{121}$$

$$J(T_{2n-1})(x) = -\frac{2\pi}{(2n-1)^2} + \frac{\pi^2}{2n-1} \cdot \sqrt{1-x^2} \cdot U_{2n-2}(x) \tag{122}$$

for all $n \geqslant 1$.

Proof. Substituting T_0 into the Eqs. (112) and (114), we obtain

$$J'(T_0)(t)\,\mathrm{d}t = \frac{-\pi^2}{\sqrt{1-t^2}}, \tag{123}$$

$$J(T_0)(-1) = \pi \cdot \int_{-1}^{1} \frac{\arccos(s)}{\sqrt{1-s^2}}\,\mathrm{d}s = \pi \cdot \int_{0}^{\pi} x\,\mathrm{d}x = \frac{\pi^3}{2}. \tag{124}$$

Now, (120) follows immediately from the combination of (123) and (124), and the trivial identity

$$J(T_0)(x) = J(T_0)(-1) + \int_{-1}^{x} J'(T_0)(t)\,\mathrm{d}t. \tag{125}$$

The proofs of (121), (122) are virtually identical to the proof of (120). $\square$

The following lemma provides explicit expressions for $L(U_n)$, with $n = 0, 1, 2, \ldots$ It is a direct analogue of Lemma 26, replacing the mapping J with the mapping L, and the polynomials T_n with the polynomials U_n. Its proof is virtually identical to that of Lemma 26.

Lemma 27. *For any* $x \in [-1, 1]$,

$$L(U_0)(x) = \frac{\pi^2}{2} \cdot (\arccos x - x \cdot \sqrt{1-x^2}) + 2\pi \cdot x - \frac{\pi^3}{4} \tag{126}$$

and

$$L(U_{2n})(x) = \frac{\pi^2}{2} \cdot \sqrt{1-x^2} \cdot \left(\frac{U_{2n-1}(x)}{2n} - \frac{U_{2n+1}(x)}{2n+2} \right) + \frac{2\pi}{2n+1} \cdot x, \tag{127}$$

$$L(U_{2n-1})(x) = \frac{\pi^2}{2} \cdot \sqrt{1-x^2} \cdot \left(\frac{U_{2n-2}(x)}{2n-1} - \frac{U_{2n}(x)}{2n+1} \right) + 2\pi \cdot \left(\frac{2n \log 2}{4n^2-1} - \frac{4n}{(4n^2-1)^2} \right) \tag{128}$$

for all $n \geqslant 1$.

We are now in a position to combine the identities (27) and (28), Lemmas 24, 26 and 27 to obtain a refined version of Lemma 24. The following theorem is one of principal analytical tools of this paper.

Theorem 28. *Suppose that for each* $n = 0, 1, 2, \ldots$, *the function* $\sigma_n \in F_I$ (*see Definition* 19) *is the solution of the equation*

$$\widetilde{H}^2(\sigma_n) = T_n. \tag{129}$$

Then

$$\sigma_0(x) = \frac{1}{\pi^3} \cdot \frac{x}{\sqrt{1-x^2}} \cdot \log\frac{1+x}{1-x} - \frac{2(\log 2 + 1)}{\pi^3 \log 2} \cdot \frac{1}{\sqrt{1-x^2}}, \tag{130}$$

$$\sigma_1(x) = \frac{1}{\pi^3} \cdot \sqrt{1-x^2} \cdot \int_{-1}^{1} \frac{U_0(t)}{t-x}\,\mathrm{d}t - \frac{2}{\pi^3} \cdot \frac{x}{\sqrt{1-x^2}} + \frac{1}{2\pi^3} \cdot \frac{1}{\sqrt{1-x^2}} \cdot \log\frac{1+x}{1-x} \tag{131}$$

and

$$\sigma_{2n}(x) = \frac{1}{\pi^3} \cdot \sqrt{1-x^2} \cdot \int_{-1}^{1} \frac{U_{2n-1}(t)}{t-x}\,\mathrm{d}t - \frac{2}{\pi^3 \log 2} \cdot \left(\frac{2n \log 2}{4n^2-1} - \frac{4n}{(4n^2-1)^2} \right) \cdot \frac{1}{\sqrt{1-x^2}}, \tag{132}$$

$$\sigma_{2n+1}(x) = \frac{1}{\pi^3} \cdot \sqrt{1-x^2} \cdot \int_{-1}^{1} \frac{U_{2n}(t)}{t-x}\,\mathrm{d}t - \frac{2}{(2n+1)\pi^3} \cdot \frac{x}{\sqrt{1-x^2}} \tag{133}$$

for all $n \geqslant 1$.

Finally, we will need the following technical lemma.

Lemma 29. *Suppose that the functions* $D_n : [-1, 1] \to \mathbb{R}$ *with* $n = 0, 1, 2, \ldots$ *are defined by the formula*

$$D_n(x) = \int_{-1}^{1} \log|x - t| \cdot \sigma_n(t)\, \mathrm{d}t, \tag{134}$$

with σ_n *defined by* (130)–(133) *above.*

Then

$$D_0(x) = \frac{1}{\pi} \cdot \sqrt{1 - x^2}, \tag{135}$$

$$D_1(x) = \frac{1}{2\pi} \cdot x \cdot \sqrt{1 - x^2} \tag{136}$$

and

$$D_n(x) = \frac{1}{2\pi} \cdot \sqrt{1 - x^2} \cdot \left(\frac{U_n(x)}{n + 1} - \frac{U_{n-2}(x)}{n - 1} \right), \tag{137}$$

for all $n \geqslant 2$.

Furthermore, for any integer $n \geqslant 2$, *there exists a polynomial* $p_{n-2}(x)$ *of degree* $n - 2$ *such that*

$$D_n(x) = (1 - x^2)^{3/2} \cdot p_{n-2}(x). \tag{138}$$

Proof. The identities (135)–(137) are a direct consequence of the identities (27) and (28), and Lemmas 26 and 27. To prove (138), we first observe that (see, for example [2]) for all $n = 0, 1, 2, \ldots$,

$$U_n(1) = n + 1, \tag{139}$$

$$U_n(-1) = (-1)^n (n + 1). \tag{140}$$

It immediately follows from 139 and 140 that

$$\frac{U_n(-1)}{n + 1} - \frac{U_{n-2}(-1)}{n - 1} = 0, \tag{141}$$

$$\frac{U_n(1)}{n + 1} - \frac{U_{n-2}(1)}{n - 1} = 0 \tag{142}$$

for any $n \geqslant 2$.

Now, we observe that the function

$$W(x) = \frac{U_n(x)}{n + 1} - \frac{U_{n-2}(x)}{n - 1} \tag{143}$$

is a polynomial of degree n, and that the points $x = \pm 1$ are the roots of W (see (141) and (142)). Therefore, there exists such a polynomial p_{n-2} of degree $n - 2$ that

$$\frac{U_n(x)}{n + 1} - \frac{U_{n-2}(x)}{n - 1} = (1 - x^2) \cdot p_{n-2}(x). \tag{144}$$

Finally, we obtain (138) by substituting (144) into (137). □

5.5. The integral equation formulation for the case of a line segment

In this section, we will combine the results in previous four sections to solve the Dirichlet problem for the line segment $I = [-1, 1]$ on the real axis. The following lemma is a direct consequence of Theorems 13 and 28, and Lemmas 23 and 29.

Lemma 30. *For any function $f \in C^2[-1, 1]$, there exists a unique solution $\sigma \in F_I$ (see Definition 19) to the equation*

$$B_I(\sigma)(x) = \pi^2 \cdot \sigma(x) + \int_{-1}^{1} \frac{\log \frac{1-x}{1-t} - \log \frac{1+x}{1+t}}{x - t} \cdot \sigma(t)\, \mathrm{d}t = f(x); \tag{145}$$

in other words, the operator B_I^{-1} is well defined if the range is restricted to the function space F_I. Furthermore, if f is orthogonal to T_0, T_1 with respect to the inner product (3), then the function $P_I(\sigma)$ can be continuously extended to $\mathbb{R}^2$.

For the cases $f = T_0$, $f = T_1$, we have the following lemma, easily verified by direct calculation.

Lemma 31.

1. *The only bounded continuous solution to the problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus I, \\ u = 1 & \text{on } I \end{cases} \tag{146}$$

is

$$u_I^0(x, y) = 1. \tag{147}$$

2. *The only bounded continuous solution to the problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus I, \\ u = x & \text{on } I \end{cases} \tag{148}$$

is

$$u_I^1(x, y) = \frac{N(x, y)}{D(x, y)}, \tag{149}$$

with the functions $N, D : \mathbb{R}^2 \to \mathbb{R}$ defined by the formulae

$$N(x, y) = \sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2}, \tag{150}$$

$$D(x, y) = \sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2} + \sqrt{\left(\sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2}\right)^2 - 4}, \tag{151}$$

respectively.

Combining Lemmas 30 and 31, we immediately obtain the following theorem.

Theorem 32. *Suppose that the function $f : [-1,1] \to \mathbb{R}$ is twice continuously differentiable. Suppose further that the function $\sigma \in F_I$ (see Definition* 19), *and the coefficients A_0, A_1 satisfy the following equations*:

$$B_I(\sigma)(x) = \pi^2 \cdot \sigma(x) + \int_{-1}^{1} \left\{ \left(\log\frac{1-x}{1-t} - \log\frac{1+x}{1+t} \right) \Big/ (x-t) \right\} \cdot \sigma(t)\,\mathrm{d}t = f(x) - A_0 - A_1 \cdot x, \tag{152}$$

$$\int_{-1}^{1} (f(x) - A_0 - A_1 \cdot x) \cdot \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = 0, \tag{153}$$

$$\int_{-1}^{1} (f(x) - A_0 - A_1 \cdot x) \cdot \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = 0. \tag{154}$$

Then the function $u : \mathbb{R}^2 \to \mathbb{R}$ defined by the formula

$$u(x,y) = P_I(\sigma)(x,y) + A_0 \cdot u_I^0(x,y) + A_1 \cdot u_I^1(x,y) \tag{155}$$

is the solution of the problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus I, \\ u = f & \text{on } I. \end{cases} \tag{156}$$

Applying Theorem 28, we can now solve the Dirichlet problem (156) via the representation (155).

Corollary 33. *Under the conditions of Theorem* 32, *the solutions to the Eqs.* (152)–(154) *are*

$$\sigma(x) = \frac{1}{\pi^3} \cdot \sqrt{1-x^2} \cdot \sum_{k=2}^{\infty} C_k \cdot \int_{-1}^{1} \frac{U_{k-1}(t)}{x-t}\,\mathrm{d}t + \frac{B_0}{\sqrt{1-x^2}} + \frac{B_1 \cdot x}{\sqrt{1-x^2}}, \tag{157}$$

$$A_0 = C_0, \tag{158}$$

$$A_1 = C_1, \tag{159}$$

where the coefficients B_0, B_1 are defined by the formulae

$$B_0 = \frac{2}{\pi^3 \cdot \log 2} \cdot \sum_{k=1}^{\infty} C_{2k} \cdot \left(\frac{2k \log 2}{4k^2 - 1} - \frac{4k}{(4k^2-1)^2} \right), \tag{160}$$

$$B_1 = \frac{2}{\pi^3} \cdot \sum_{k=1}^{\infty} \frac{C_{2k+1}}{2k+1}, \tag{161}$$

respectively, and C_k $(k = 0, 1, 2, \ldots)$ *are the Chebyshev coefficients of f given by* (9) *and* (10).

Remark 34. It immediately follows from Lemma 29 that the function $P_I(\sigma)$ with σ given by (157) has an explicit expression

$$P_I(\sigma)(x,y) = \int_{-1}^{1} \frac{(x-s)^2 - y^2}{((x-s)^2 + y^2)^2} \cdot D(\sigma)(s)\,\mathrm{d}s, \tag{162}$$

for any $(x,y) \in \mathbb{R}^2 \setminus I$, with the function $D(\sigma) : [-1,1] \to \mathbb{R}$ defined by the formula

$$D(\sigma)(x) = \frac{1}{2\pi} \cdot \sqrt{1-x^2} \cdot \sum_{k=2}^{\infty} C_k \cdot \left(\frac{U_{k-2}(x)}{k-1} - \frac{U_k(x)}{k+1} \right). \tag{163}$$

Finally, we will need the following lemma.

Lemma 35. *Suppose that the operator S is defined by the formula*

$$S(\eta)(x) = D(B_I^{-1}(\eta))(x) = \int_{-1}^{1} \log|x-t| \cdot B_I^{-1}(\eta)(t)\,\mathrm{d}t, \tag{164}$$

with the operator B_I defined in (96). *Then S is a bounded linear operator from $C[-1,1]$ to $C[-1,1]$.*

Proof. By Lemma 29, we have

$$S(T_0)(x) = -\frac{1}{\pi} \cdot \sqrt{1-x^2}, \tag{165}$$

$$S(T_1)(x) = -\frac{1}{2\pi} \cdot x \cdot \sqrt{1-x^2}, \tag{166}$$

and

$$S(T_n)(x) = -\frac{1}{2\pi} \cdot \sqrt{1-x^2} \cdot \left(\frac{U_n(x)}{n+1} - \frac{U_{n-2}(x)}{n-1} \right), \tag{167}$$

for all $n \geqslant 2$. Substituting (5) into (167), we obtain

$$S(T_n)(x) = -\frac{1}{2\pi} \cdot \left(\frac{\sin((n+1)\arccos(x))}{n+1} - \frac{\sin((n-1)\arccos(x))}{n-1} \right), \tag{168}$$

for all $n \geqslant 2$. Utilizing the trivial fact that $|\sin(u)| \leqslant 1$ for any real number u, we have

$$\|S(T_n)\|_\infty \leqslant \frac{2}{\pi} \cdot \frac{1}{n+1}, \tag{169}$$

for all $n = 0, 1, 2, \ldots$ Now, any function $\varphi \in C^2[-1,1]$ can be expanded into a Chebyshev series

$$\varphi(x) = \sum_{n=0}^{\infty} C_n \cdot T_n(x), \tag{170}$$

and by Parseval's identity,

$$\sum_{n=0}^{\infty} C_n^2 = \int_{-1}^{1} \frac{\varphi(x)^2}{\sqrt{1-x^2}}\,\mathrm{d}x \leqslant \pi \cdot \|\varphi\|_\infty^2. \tag{171}$$

Applying Schwarz's inequality, we have

$$\|S(\varphi)\|_\infty \leqslant \sum_{n=0}^{\infty} |C_n| \cdot \|S(T_n)\|_\infty \leqslant \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot |C_n| \leqslant \frac{2}{\pi} \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \right)^{1/2} \cdot \left(\sum_{n=0}^{\infty} C_n^2 \right)^{1/2} \leqslant 2\|\varphi\|_\infty. \tag{172}$$

Since $C^2[-1,1]$ is dense in $C[-1,1]$, S is bounded from $C[-1,1]$ to $C[-1,1]$. $\square$

6. Analytical apparatus III: open surface problem on a general curve

6.1. The integral operator P_γ

In this section, we consider the case of a general curve. We assume that $\gamma : [-1, 1] \to \mathbb{R}^2$ is a sufficiently smooth "open" curve with the parametrization (71). The image of γ is denoted by Γ. We will consider the operator $P_\gamma : F_I \to C^2(\mathbb{R}^2 \setminus \Gamma)$ defined by the formula

$$\begin{aligned} P_\gamma(\sigma)(x) &= \int_{-1}^{1} K_\gamma(x,t) \cdot \sigma(t)\,\mathrm{d}t \\ &= \frac{L^2}{4} \cdot \int_{-1}^{1} \left(\frac{\partial^2 G_{\gamma(s)}(x)}{\partial N(s)^2} - c(s)\frac{\partial G_{\gamma(s)}(x)}{\partial N(s)} \right) \cdot \left(\int_{-1}^{1} \log|s-t|\sigma(t)\,\mathrm{d}t \right) \mathrm{d}s, \end{aligned} \tag{173}$$

with L the arc length of γ. The following lemma provides an explicit expression for the kernel K_γ. Its proof is virtually identical to that of Lemma 20.

Lemma 36. *For any $\sigma \in F_I$ (see Definition 19),*

$$K_\gamma(x,t) = \int_{-1}^{1} \frac{\partial G_{\gamma(s)}(x)}{\partial T(s)} \cdot \frac{1}{s-t}\,\mathrm{d}s, \tag{174}$$

for any $x \in \mathbb{R}^2 \setminus \Gamma$ and $t \in (-1, 1)$, with the integral in (174) intepreted in the principal value sense.

6.2. The boundary integral operator B_γ

We will then define the integral operator $B_\gamma : F_I \to L^1[-1, 1]$ by the formula

$$B_\gamma(\sigma)(t) = \lim_{h\to 0} P_\gamma(\sigma)(\gamma(t) + h \cdot N(t)) = \lim_{h\to 0} \int_{-1}^{1} K_\gamma(\gamma(t) + h \cdot N(t), s) \cdot \sigma(s)\,\mathrm{d}s. \tag{175}$$

The following lemma is a direct consequence of Lemmas 12 and 21; it provides an explicit expression for B_γ.

Lemma 37. *For any $t \in (-1, 1)$,*

$$B_\gamma(\sigma)(t) = \pi^2 \cdot \sigma(t) + \int_{-1}^{1} K_\gamma^b(t,s) \cdot \sigma(s)\,\mathrm{d}s, \tag{176}$$

with the kernel $K_\gamma^b : (-1, 1) \times (-1, 1) \to \mathbb{R}$ given by the formula

$$K_\gamma^b(t,s) = \int_{-1}^{1} \frac{\partial G_{\gamma(x)}(\gamma(t))}{\partial T(x)} \cdot \frac{1}{x-s}\,\mathrm{d}x, \tag{177}$$

with the integral in (177) intepreted in the principal value sense.

6.3. The integral equation formulation for the case of a general curve

Similarly to the operator B_I defined in (96), the kernel K_γ^b of B_γ is strongly singular at the end-points. Therefore, if the solution of the Dirichlet problem (69) and (70) is represented by the function $P_\gamma(\sigma)$ on $\mathbb{R}^2 \setminus \Gamma$, then (70) will lead to a boundary integral equation

$$B_\gamma(\sigma)(t) = f(t), \tag{178}$$

which is *not* of the second kind. Because of the obvious similarity of the operators B_I, B_γ, it is natural to consider the operator $\widetilde{P}_\gamma : C[-1,1] \to C^2(\mathbb{R}^2 \setminus \Gamma)$ defined by the formula

$$\widetilde{P}_\gamma(\eta)(x) = P_\gamma \circ B_I^{-1}(\eta)(x). \tag{179}$$

Obviously, $\widetilde{P}_\gamma(\eta)$ is not defined when $x \in \Gamma$, and we will define the operator $\widetilde{B}_\gamma : C[-1,1] \to C[-1,1]$ by the formula

$$\widetilde{B}_\gamma(\eta)(t) = \lim_{x \to \gamma(t)} \widetilde{P}_\gamma(\eta)(x) = B_\gamma \circ B_I^{-1}(\eta)(t). \tag{180}$$

The following theorem is one of principal results of the paper; it states that $\widetilde{B}_\gamma$ is a *second kind* integral operator when restricted to continuous functions, and is an immediate consequence of Lemmas 15 and 35.

Theorem 38. *Suppose that* $\gamma : [-1,1] \to \mathbb{R}^2$ *is a sufficiently smooth* "open" curve with the parametrization (71). *Suppose further that the operator* $\widetilde{R}_\gamma : C[-1,1] \to C[-1,1]$ *is defined by the formula*

$$\widetilde{R}_\gamma(\sigma)(x) = \int_{-1}^{1} \widetilde{r}(x,t) \cdot \sigma(t)\,\mathrm{d}t, \tag{181}$$

with the function $\widetilde{r} : [-1,1] \times [-1,1] \to \mathbb{R}$ *defined by the formula*

$$\widetilde{r}(x,t) = \frac{L^2}{4} \cdot \left(-\frac{2\langle N(t), \gamma(x) - \gamma(t)\rangle^2}{\|\gamma(x) - \gamma(t)\|^4} + \frac{1}{\|\gamma(x) - \gamma(t)\|^2} \right) + \frac{L^2 \cdot c(t)}{4} \cdot \frac{\langle N(t), \gamma(x) - \gamma(t)\rangle}{\|\gamma(x) - \gamma(t)\|^2} - \frac{1}{(x-t)^2}, \tag{182}$$

for all $x \neq t$, *and by the formula*

$$\widetilde{r}(t,t) = \frac{L^2 \cdot c(t)^2}{48}, \tag{183}$$

for all $x = t$, *with* L *the arc length of* γ, *and* $c(t)$ *the curvature of* γ *at the point* $\gamma(t)$. *Then,*

$$\widetilde{B}_\gamma(\eta)(t) = (I + M)(\eta)(t), \tag{184}$$

with $I : C[-1,1] \to C[-1,1]$ *the identity operator, and* $M : C[-1,1] \to C[-1,1]$ *a compact operator defined by the formula*

$$M(\eta)(t) = (B_\gamma - B_I) \circ B_I^{-1}(\eta)(t) = \widetilde{R}_\gamma \circ S(\eta)(t), \tag{185}$$

with the operators $\widetilde{R}_\gamma, S : C[-1,1] \to C[-1,1]$ *defined by* ((181), (164)–(167)), *respectively.*

Remark 39. It immediately follows from the combination of (59) and (182) that the operator $\widetilde{R}_\gamma$ is related to R_γ defined in Lemma 15 by the formula

$$\widetilde{R}_\gamma(\widetilde{\sigma})(x) = \frac{L}{2} \cdot R_\gamma(\sigma)\left(\frac{L}{2}(x+1)\right), \tag{186}$$

with $\widetilde{\sigma}(t) = \sigma(\frac{L}{2}(t+1))$, and the function $\widetilde{r}$ is related to the function r defined in (59) by the formula

$$\widetilde{r}(x,t) = \frac{L^2}{4} \cdot r\left(\frac{L}{2}(x+1), \frac{L}{2}(t+1)\right). \tag{187}$$

The function $\widetilde{P}_\gamma(\eta)$ cannot, in general, be continuously extended to the whole plane $\mathbb{R}^2$, unless the density η satisfies certain additional conditions. The following lemma is a direct consequence of Theorems 13 and 28, and Lemmas 23 and 29.

Lemma 40. *Suppose that the function $\eta \in C[-1,1]$ is orthogonal to T_0 and T_1 with respect to the inner product* (3). *Then $\widetilde{P}_\gamma(\eta)$ can be continuously extended to $\mathbb{R}^2$.*

Lemma 40 above shows that the solution of the problem (69) and (70) cannot be represented by the function $\widetilde{P}_\gamma(\eta)$ alone. Indeed, $\widetilde{P}_\gamma(\eta)(x)$ decays at infinity like $1/|x|$, whereas Theorem 17 only requires that the solution of the problem (69) and (70) be bounded at infinity. Suppose now that we can find two functions u_γ^0, u_γ^1 in S_γ (see (68)) such that the following condition holds:

$$\det\begin{pmatrix} \langle\eta_0, T_0\rangle & \langle\eta_0, T_1\rangle \\ \langle\eta_1, T_0\rangle & \langle\eta_1, T_1\rangle \end{pmatrix} \neq 0, \tag{188}$$

with η_0, η_1 the solutions to the equations

$$\widetilde{B}_\gamma(\eta_0)(t) = u_\gamma^0(\gamma(t)), \tag{189}$$

$$\widetilde{B}_\gamma(\eta_1)(t) = u_\gamma^1(\gamma(t)), \tag{190}$$

respectively, and the inner product in (188) defined by (3). Then the solution of the problem (69) and (70) can be represented by the formula

$$u(x) = \widetilde{P}_\gamma(\eta)(x) + A_0 \cdot u_\gamma^0(x) + A_1 \cdot u_\gamma^1(x), \tag{191}$$

so that the density η, while satisfying the boundary integral equation

$$\widetilde{B}_\gamma(\eta)(t) = f(t) - A_0 \cdot u_\gamma^0(\gamma(t)) - A_1 \cdot u_\gamma^1(\gamma(t)), \tag{192}$$

is also orthogonal to T_0 and T_1. The following lemma provides such two functions indirectly; it describes a single-layer-potential representation for the functions $\widetilde{P}_\gamma(T_n)$ $(n = 2, 3, \ldots .)$.

Lemma 41. *Suppose that $\gamma : [-1,1] \to \mathbb{R}^2$ is a sufficiently smooth* "open" curve with the parametrization (71). *Then for any $n = 2, 3, \ldots,$*

$$\widetilde{P}_\gamma(T_n)(x) = -\frac{n}{\pi} \cdot \int_{-1}^{1} G_{\gamma(t)}(x) \cdot \frac{T_n(t)}{\sqrt{1-t^2}}\,\mathrm{d}t = -\frac{n}{\pi} \cdot \int_{-1}^{1} \log|x - \gamma(t)| \cdot \frac{T_n(t)}{\sqrt{1-t^2}}\,\mathrm{d}t, \tag{193}$$

for any $x \notin \Gamma$.

Proof. Combining (179), (173), (164), we have the identity

$$\widetilde{P}_\gamma(\eta)(x) = \frac{L^2}{4} \cdot \int_{-1}^{1} \left(\frac{\partial^2 G_{\gamma(t)}(x)}{\partial N(t)^2} - c(t) \frac{\partial G_{\gamma(t)}(x)}{\partial N(t)} \right) \cdot S(\eta)(t)\,\mathrm{d}t, \tag{194}$$

for an arbitrary $\eta \in C[-1,1]$. In particular,

$$\widetilde{P}_\gamma(T_n)(x) = \frac{L^2}{4} \cdot \int_{-1}^{1} \left(\frac{\partial^2 G_{\gamma(t)}(x)}{\partial N(t)^2} - c(t) \frac{\partial G_{\gamma(t)}(x)}{\partial N(t)} \right) \cdot S(T_n)(t)\,\mathrm{d}t. \tag{195}$$

Since the function $G_{\gamma(t)}(x)$ satisfies the Laplace equation for all $x \neq \gamma(t)$, applying (33) to $G_{\gamma(t)}$ and carrying out elementary analytic manipulations, we obtain the identity

$$\frac{L^2}{4} \cdot \left(\frac{\partial^2 G_{\gamma(t)}(x)}{\partial N(t)^2} - c(t) \frac{\partial G_{\gamma(t)}(x)}{\partial N(t)} \right) = -\frac{\partial^2 G_{\gamma(t)(x)}}{\partial t^2}, \tag{196}$$

and substitution of (196), (167) into (195) yields the identity

$$\widetilde{P}_\gamma(T_n)(x) = \frac{1}{2\pi} \int_{-1}^{1} \frac{\partial^2 G_{\gamma(t)(x)}}{\partial t^2} \cdot \sqrt{1-t^2} \cdot \left(\frac{U_n(t)}{n+1} - \frac{U_{n-2}(t)}{n-1} \right) \mathrm{d}t. \tag{197}$$

Now, we obtain (193) by integrating by parts twice the right-hand side of (197). □

The following lemma is an immediate consequence of Lemma 41 and the well-known fact that the functions $u_\gamma^n : \mathbb{R}^2 \to \mathbb{R}$ $(n = 0, 1, 2, \ldots)$ defined by the formulae

$$u_\gamma^0(x) = 1, \tag{198}$$

$$u_\gamma^n(x) = \int_{-1}^{1} \log|x - \gamma(t)| \cdot \frac{T_n(t)}{\sqrt{1-t^2}} \mathrm{d}t, \quad n = 1, 2, \ldots \tag{199}$$

form a complete basis for the space S_γ (see, for example [15]).

Lemma 42. *Suppose that* $\gamma : [-1, 1] \to \mathbb{R}^2$ *is a sufficiently smooth* "open" curve with the parametrization (71). *Then the functions* u_γ^0, u_γ^1 *defined by* (198) *and* (199) *satisfy the condition* (188)–(190).

Finally, we summarize our analysis for the case of a general curve by the following theorem.

Theorem 43. *Suppose that* $\gamma : [-1, 1] \to \mathbb{R}^2$ *is a sufficiently smooth* "open" curve with the parametrization (71), *and that the function* $f : [-1, 1] \to \mathbb{R}$ *is twice continuously differentiable. Suppose further that the function* $\eta : [-1, 1] \to \mathbb{R}$, *and the coefficients* A_0, A_1 *satisfy the equations*

$$\widetilde{B}_\gamma(\eta)(t) = (I + \widetilde{R}_\gamma \circ S)(\eta)(t) = f(t) - A_0 \cdot u_\gamma^0(\gamma(t)) - A_1 \cdot u_\gamma^1(\gamma(t)), \tag{200}$$

$$\int_{-1}^{1} \eta(t) \cdot \frac{1}{\sqrt{1-t^2}} \mathrm{d}t = 0, \tag{201}$$

$$\int_{-1}^{1} \eta(t) \cdot \frac{t}{\sqrt{1-t^2}} \mathrm{d}t = 0, \tag{202}$$

with I *the identity operator, and the operators* $\widetilde{R}_\gamma, S : C[-1, 1] \to C[-1, 1]$ *defined by* (181) *and* (164), *respectively. Then the function* $u : \mathbb{R}^2 \to \mathbb{R}$ *defined by the formula*

$$u(x) = \widetilde{P}_\gamma(\eta)(x) + A_0 \cdot u_\gamma^0(x) + A_1 \cdot u_\gamma^1(x) \tag{203}$$

is the solution of the problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus \Gamma, \\ u = f & \text{on } \Gamma, \end{cases} \tag{204}$$

in (203), *the operator* $\widetilde{P}_\gamma : C[-1, 1] \to C(\mathbb{R}^2)$ *is defined by* (179), (173), (145), *and the functions* u_γ^0, u_γ^1 *are defined by* (198) *and* (199) *respectively.*

Remark 44. For the case of several open curves $\Gamma = \sum_{i=1}^{m} \gamma_i$, the following modifications should be made. Instead of (203), the function u will be given by the formula

$$u(x) = \sum_{i=1}^{m} \left\{ \widetilde{P}_{\gamma_i}(\eta_i)(x) + A_0^i \cdot u_{\gamma_i}^0(x) + A_1^i \cdot u_{\gamma_i}^1(x) \right\} + C, \tag{205}$$

with C a real number to be determined, and the functions $u_{\gamma_i}^n$ $(i = 1, \ldots, m;\ n = 0, 1)$ defined by the formula

$$u_{\gamma_i}^n(x) = \int_{-1}^{1} \log|x - \gamma_i(t)| \cdot \frac{T_n(t)}{\sqrt{1-t^2}}\, \mathrm{d}t. \tag{206}$$

The functions η_i, and the coefficients A_0^i, A_1^i, C are determined as the solution of the system of equations

$$\widetilde{B}_{\gamma_i}(\eta_i)(t) = (I + \widetilde{R}_{\gamma_i} \circ S)(\eta_i)(t) = f_i(t) - \sum_{j=1}^{m} \left\{ A_0^j \cdot u_{\gamma_j}^0(\gamma_i(t)) - A_1^j \cdot u_{\gamma_j}^1(\gamma_i(t)) \right\} - C, \tag{207}$$

$$\int_{-1}^{1} \eta_i(t) \cdot \frac{1}{\sqrt{1-t^2}}\, \mathrm{d}t = 0, \tag{208}$$

$$\int_{-1}^{1} \eta_i(t) \cdot \frac{t}{\sqrt{1-t^2}}\, \mathrm{d}t = 0, \tag{209}$$

$$\sum_{i=1}^{m} A_0^i = 0. \tag{210}$$

Clearly, the functions $u_{\gamma_i}^0$ defined by (206) are linearly independent; the constraint (210) and the constant term C are introduced so that the function u is bounded at infinity.

7. Numerical algorithm

In this section, we construct a rudimentary numerical algorithm for the solution of the Dirichlet problem (69) and (70) via the Eqs. (200)–(202). Since the construction of the matrix and the solver of the resulting linear system are direct, the algorithm requires $O(N^3)$ work and $O(N^2)$ storage, with N the number of nodes on the boundary. While standard acceleration techniques (such as the Fast Multipole Method, etc.) could be used to improve these estimates, no such acceleration was performed, since the purpose of this section (as well as the following one) is to demonstrate the stability of the integral formulation and the convergence rate of a very simple discretization scheme.

By Theorem 43, the equations to be solved are (200)–(202), where the unknowns are the function η, and two real numbers A_0, A_1. To solve (200)–(202) numerically, we discretize the boundary into N Chebyshev nodes and approximate the unknown density η by a finite Chebyshev series of the first kind,

$$\eta(t) \simeq \sum_{k=0}^{N-1} C_k \cdot T_k(t), \tag{211}$$

with the coefficients C_k $(k = 0, \ldots, N-1)$ to be determined. In order to discretize (200), we start with observing that by (165)–(167), the action of the operator S on the function η is described via the formula

$$S(\eta)(x) = \sum_{k=0}^{N-1}\left(\sum_{j=0}^{N-1} B_{kj}\cdot C_j\right)\cdot\frac{2}{\pi}\cdot U_k(x)\cdot\sqrt{1-x^2}, \tag{212}$$

where the matrix $B = (B_{kj})$ $(k, j = 0, \ldots, N-1)$ is given by the formulae

$$\begin{cases} B_{00} = -\frac{1}{2}, & \\ B_{kk} = -\frac{1}{4k}, & 1 \leqslant k \leqslant N-1, \\ B_{k,k+2} = \frac{1}{4k}, & 0 \leqslant k \leqslant N-3, \\ B_{kj} = 0, & \text{otherwise}. \end{cases} \tag{213}$$

In other words, given a function η expressed as a Chebyshev series of the first kind, (212) expresses $S(\eta)$ as a Chebyshev series of the second kind. Now, it is natural to approximate the operator $\widetilde{R}_\gamma$ by an expression converting functions of the form

$$\sum_{k=0}^{N-1} \alpha_k \cdot U_k(t) \tag{214}$$

into functions of the form

$$\sum_{k=0}^{N-1} \beta_k \cdot T_k(x), \tag{215}$$

with the product $\widetilde{R}_\gamma \circ S$ converting expressions of the form (215) into expressions of the same form. Thus, we approximate the kernel $\widetilde{r}(x,t)$ (see (182)) of the operator $\widetilde{R}_\gamma$ with an expression of the form

$$\widetilde{r}(x,t) \simeq \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} K_{ij}\cdot T_i(x)\cdot U_j(t). \tag{216}$$

Clearly, the coefficients K_{ij} have to be determined numerically, since the curve Γ is user-specified, and is unlikely to have a convenient analytical expression. Thus, we obtain the coefficients K_{ij} by first constructing the $N\times N$ matrix $R = (\widetilde{r}(x_i, t_j))$ $(i, j = 0, 1, \ldots, N-1)$ with $x_i (i = 0, 1, \ldots, N-1)$ the Chebyshev nodes defined by (4) and $t_j (j = 0, \ldots, N-1)$ the Chebyshev nodes of the second kind defined by (7), then converting R into the matrix $K = (K_{ij})(i, j = 0, 1, \ldots, N-1)$ by the formula

$$K = U\cdot R\cdot V, \tag{217}$$

with $N\times N$ matrices $U = (U_{ij})$, $V = (V_{ij})$ defined by the formulae

$$\begin{cases} U_{0j} = \frac{1}{N}\cdot T_0(x_j), & j = 0, 1, \ldots, N-1, \\ U_{ij} = \frac{2}{N}\cdot T_i(x_j), & i = 1, \ldots, N-1, \quad j = 0, 1, \ldots, N-1, \end{cases} \tag{218}$$

$$V_{ij} = \frac{2}{N+1}\cdot\sin^2\left(\frac{(N-i)\cdot\pi}{N+1}\right)\cdot U_j(t_i), \quad i, j = 0, 1, \ldots, N-1, \tag{219}$$

respectively. We then approximate the prescribed Dirichlet data f by its Chebyshev approximation of order $N-1$

$$f(t) \simeq \sum_{k=0}^{N-1}\hat{f}_k\cdot T_k(t), \tag{220}$$

where the coefficients $\hat{f}_k$ can be obtained by first evaluating f at Chebyshev nodes x_i, then applying to it the matrix U defined by (218), i.e.,

$$\hat{f}_k = \sum_{i=0}^{N-1} U_{ki} \cdot f(x_i). \tag{221}$$

Similarly, we approximate the function u_γ^1 (see (199)) with an expression of the form

$$u_\gamma^1(\gamma(t)) \simeq \sum_{k=0}^{N-1} \hat{u}_k \cdot T_k(t), \tag{222}$$

with the coefficients $\hat{u}_k$ defined by the formula

$$\hat{u}_k = \sum_{i=0}^{N-1} U_{ki} \cdot u_\gamma^1(\gamma(x_i)), \tag{223}$$

with x_i the Chebyshev nodes defined by (4). Combining (212), (216), (221), and (222), we discretize (200) into the equation

$$\widetilde{A} \cdot \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{N-1} \end{pmatrix} + A_0 \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + A_1 \cdot \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_{N-1} \end{pmatrix} = \begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{N-1} \end{pmatrix}, \tag{224}$$

with $N \times N$ matrix $\widetilde{A}$ defined by the formula

$$\widetilde{A} = I_N + K \cdot B, \tag{225}$$

with I_N the $N \times N$ identity matrix. Furthermore, (201) and (202) lead to the equations

$$C_0 = 0, \tag{226}$$

$$C_1 = 0. \tag{227}$$

Finally, combining (224), (226), (227), we obtain the following linear system of dimension $N+2$ to be solved

$$\begin{pmatrix} 1\ 0\ 0 \ldots 0 & 0 & 0 \\ 0\ 1\ 0 \ldots 0 & 0 & 0 \\ & 1 & \hat{u}_0 \\ \widetilde{A} & 0 & \hat{u}_1 \\ & \vdots & \vdots \\ & 0 & \hat{u}_{N-1} \end{pmatrix} \cdot \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{N-1} \\ A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{N-1} \end{pmatrix}. \tag{228}$$

Remark 45. Having solved (228) with any standard solver (we used DGECO from LINPACK), we can compute the solution of the Problem (69) and (70) at any point in $\mathbb{R}^2$ via (203).

Remark 46. The algorithm can be generalized to the case when the boundary consists of several disjoint open curves, and the generalization is straightforward (see Remark 44).

8. Numerical examples

A FORTRAN code has been written implementing the algorithm described in the preceding section. In this section, we demonstrate the performance of the scheme with several numerical examples. We consider the problem in electrostatics: the boundary is made of conductor and grounded, the electric field incident on the boundary is generated by the sources outside the boundary; that is to say, there are three fields present: the incident field u_{i}, the reflected field u_{r}, and the total field $u_{\mathrm{t}} = u_{\mathrm{i}} + u_{\mathrm{r}}$, where $u_t = 0$ on the boundary, and $u_{\mathrm{r}} = -u_{\mathrm{i}}$ on the boundary and is harmonic elsewhere. For these examples, we plot the equipotential lines of the total field and present tables showing the convergence rate of the algorithm by computing the errors of the reflected field.

Remark 47. In the examples below, the problems to be solved via the procedure of the preceding section have no simple analytical solution. Thus, we tested the accuracy of our procedure by evaluating our solution via the formula (203) at a large number M of nodes on the boundary Γ (in our experiments, we always used $M = 4000$), and comparing it with the analytically evaluated right-hand side. We did not need to verify the fact that our solutions satisfy the Laplace equation, since this follows directly from the representation (203).

In each of those tables, the first column contains the total number N of nodes in the discretization of each curve. The second column contains the condition number of the linear system. The third column contains the relative L^2 error of the numerical solution as compared with the analytically evaluated Dirichlet data on the boundary. The fourth column contains the maximum absolute error on the boundary. In the last two columns, we list the errors of the numerical solution as compared with the numerical solution with twice the number of nodes, where the solution is evaluated at 4000 equispaced points on a circle of radius 1.4 centered at the origin; the fifth column contains the relative L^2 error, and the sixth column contains the maximum absolute error.

Example 1. In this example, the boundary is the line segment parametrized by the formula

$$\begin{cases} x(t) = t, \\ y(t) = -0.2, \end{cases} \quad -1 \leqslant t \leqslant 1. \tag{229}$$

The Dirichlet data are generated by a unit charge at (0,0). The numerical results are shown in Table 1. The source, curve and equipotential lines are plotted in Fig. 1.

Example 2. In this example, the boundary is an elliptic arc parametrized by the formula

$$\begin{cases} x(t) = 0.8\cos(t), \\ y(t) = 0.5\sin(t) + 0.25, \end{cases} \quad -\pi \leqslant t \leqslant 0. \tag{230}$$

Table 1
Numerical results for Example 1

N	K	$E^2(\Gamma)$	$E^\infty(\Gamma)$	$E^2(u)$	$E^\infty(u)$
4	$0.524E+01$	$0.288E+00$	$0.607E+00$	$0.513E-01$	$0.590E-01$
8	$0.450E+01$	$0.703E-01$	$0.178E+00$	$0.613E-02$	$0.686E-02$
16	$0.388E+01$	$0.759E-02$	$0.212E-01$	$0.133E-03$	$0.146E-03$
32	$0.344E+01$	$0.165E-03$	$0.486E-03$	$0.115E-06$	$0.126E-06$
64	$0.318E+01$	$0.147E-06$	$0.446E-06$	$0.146E-12$	$0.164E-12$
128	$0.303E+01$	$0.252E-12$	$0.839E-12$	$0.250E-13$	$0.265E-13$

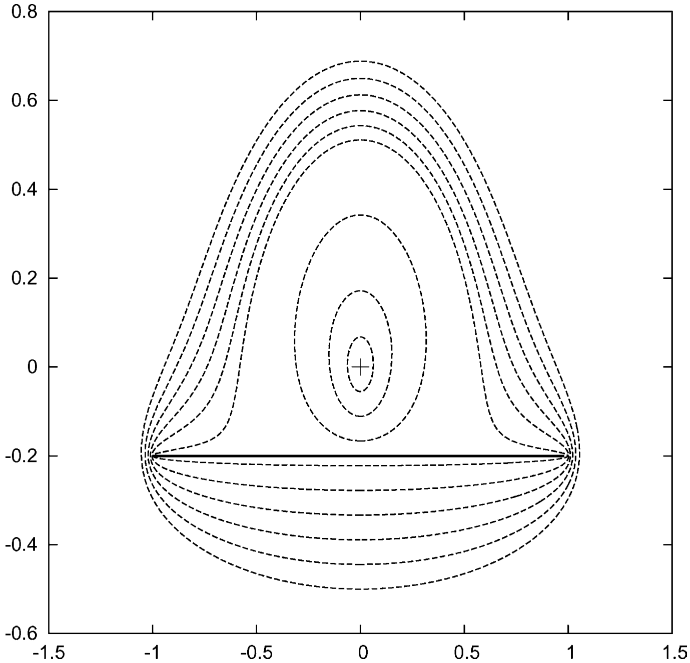

Fig. 1. Source, curve, and equipotential lines for Example 1.

Table 2
Numerical results for Example 2

N	K	$E^2(\Gamma)$	$E^\infty(\Gamma)$	$E^2(u)$	$E^\infty(u)$
4	$0.513E+01$	$0.180E+00$	$0.124E+00$	$0.343E-01$	$0.166E-01$
8	$0.461E+01$	$0.722E-01$	$0.554E-01$	$0.668E-02$	$0.333E-02$
16	$0.399E+01$	$0.103E-01$	$0.833E-02$	$0.155E-03$	$0.773E-04$
32	$0.352E+01$	$0.230E-03$	$0.187E-03$	$0.855E-07$	$0.426E-07$
64	$0.316E+01$	$0.128E-06$	$0.105E-06$	$0.475E-13$	$0.201E-13$
128	$0.301E+01$	$0.141E-12$	$0.134E-12$	$0.272E-13$	$0.102E-13$

The Dirichlet data are generated by one positive charge of unit strength at (0,0) and another negative charge of unit strength at (0,−0.5). The numerical results are shown in Table 2. The sources, curve, and equipotential lines are plotted in Fig. 2.

Example 3. In this example, the boundary is a spiral parametrized by the formula

$$\begin{cases} x(t) = t\cos(3.3t) - 0.1, \\ y(t) = t\sin(3.3t), \end{cases} \quad 0.2 \leqslant t \leqslant 1.2. \tag{231}$$

The Dirichlet data are generated by a unit charge at (0,0). The numerical results are shown in Table 3. The source, curve, and equipotential lines are plotted in Fig. 3.

Example 4. In this example, we consider the case of several open curves. The boundary consists of three elliptic arcs parametrized by the formulae

$$\begin{cases} x_1(t) = 1.1\cos(t) - 1, \\ y_1(t) = \sin(t) + 0.5, \end{cases} \quad -\frac{\pi}{12} \leqslant t \leqslant \frac{\pi}{4}, \tag{232}$$

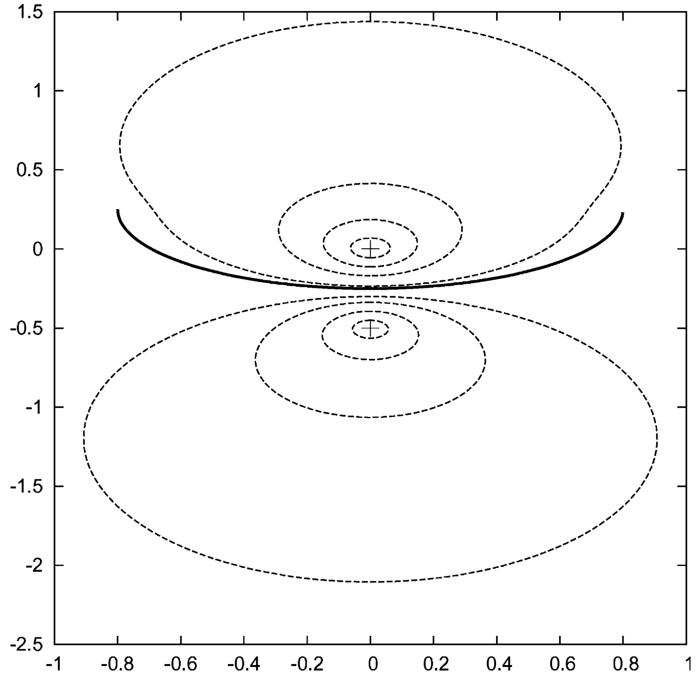

Fig. 2. Sources, curve, and equipotential lines for Example 2.

Table 3
Numerical results for Example 3

N	K	$E^2(\Gamma)$	$E^\infty(\Gamma)$	$E^2(u)$	$E^\infty(u)$
8	$0.325E+02$	$0.215E-01$	$0.323E-01$	$0.478E+00$	$0.426E+00$
16	$0.579E+01$	$0.549E-03$	$0.986E-03$	$0.658E-01$	$0.820E-01$
32	$0.478E+01$	$0.211E-05$	$0.317E-05$	$0.149E-02$	$0.194E-02$
64	$0.424E+01$	$0.987E-11$	$0.122E-10$	$0.350E-06$	$0.453E-06$
128	$0.392E+01$	$0.861E-13$	$0.520E-12$	$0.127E-12$	$0.119E-12$
256	$0.374E+01$	$0.138E-12$	$0.139E-11$	$0.139E-12$	$0.123E-12$

$$\begin{cases} x_2(t) = 1.1\cos(t), \\ y_2(t) = \sin(t) - 1.2, \end{cases} \quad \frac{7\pi}{12} \leqslant t \leqslant \frac{11\pi}{12}, \tag{233}$$

$$\begin{cases} x_3(t) = 1.1\cos(t) + 1, \\ y_3(t) = \sin(t) + 0.5, \end{cases} \quad -\frac{3\pi}{4} \leqslant t \leqslant -\frac{5\pi}{12}. \tag{234}$$

The Dirichlet data are generated by a unit charges at (0,0). The numerical results are shown in Table 4, where N is the number of nodes on each curve. The source, curves, and equipotential lines are plotted in Fig. 4.

Remark 48. The above examples illustrate the superalgebraic convergence of the scheme for smooth data and curves (see Remark 16 in Section 2.6). The number of nodes needed depends on the complexity of the underlying geometry and the smoothness of the prescribed data. The condition number of the resulting linear system is usually very low.

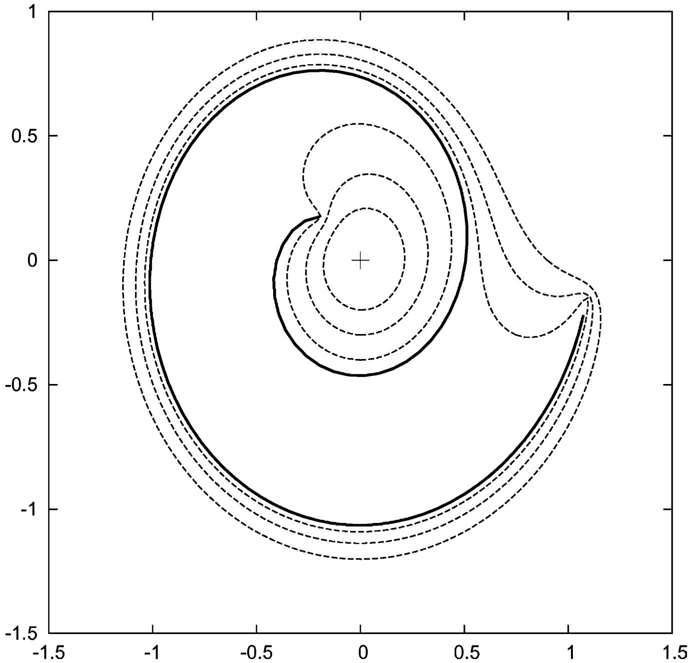

Fig. 3. Source, curve, and equipotential lines for Example 3.

Table 4
Numerical results for Example 4

N	K	$E^2(\Gamma)$	$E^\infty(\Gamma)$	$E^2(u)$	$E^\infty(u)$
4	$0.845E+01$	$0.113E-01$	$0.228E-01$	$0.493E-03$	$0.117E-02$
8	$0.754E+01$	$0.126E-03$	$0.269E-03$	$0.159E-05$	$0.108E-04$
16	$0.689E+01$	$0.173E-07$	$0.390E-07$	$0.656E-10$	$0.452E-09$
32	$0.649E+01$	$0.443E-12$	$0.196E-11$	$0.950E-13$	$0.113E-12$
64	$0.627E+01$	$0.658E-13$	$0.295E-12$	$0.492E-14$	$0.433E-14$
128	$0.615E+01$	$0.880E-13$	$0.356E-12$	$0.968E-14$	$0.971E-14$

9. Conclusions and generalizations

We have presented a stable second kind integral equation formulation for the Dirichlet problem for the Laplace equation in two dimensions, with the boundary condition specified on a curve (consisting of one or more separate segments). The resulting numerical algorithm converges superalgebraically if both the boundary data and the curves are smooth. Obviously, the combination of the Fast Multipole Method (see, for example, [7]) and any standard iterative solver yields an $O(N)$ algorithm, with N the number of nodes on the boundary.

The extensions of the scheme of this paper to other boundary conditions (such as Neumann condition, Robin condition, etc.) specified on an open curve Γ in $\mathbb{R}^2$ are fairly straightforward. For the Neumann problem, representing the solution in the form of a double layer potential, one obtains a hypersingular integral equation on Γ. Its subsequent preconditioning by a single layer potential yields a second kind integral equation (SKIE). For a Robin problem, one obtains an SKIE formulation by representing the

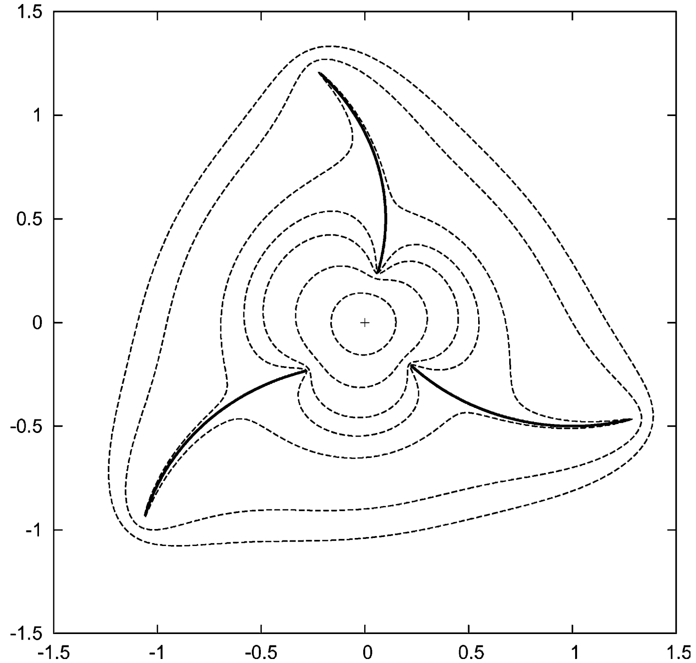

Fig. 4. Source, curves, and equipotential lines for Example 4.

solution via an appropriate linear combination of single and double layer potentials, with a further preconditioning by a single layer potential. Furthermore, the approach of this paper can be applied almost without modification to elliptic PDEs other than the Laplace equation (such as Helmholtz equation, Yukawa equaiton, etc.). Indeed, the Green's function for any such equation has the form

$$G(x,y) = \phi(x,y) \cdot \log(||x-y||) + \psi(x,y), \tag{235}$$

with ϕ, ψ a pair of smooth functions, and $\phi(0,0) = 1/(2\pi)$ (see, for example [4]). When the procedure of Section 6 of this paper is applied to a Green's function of the form (235), the result is virtually identical to that obtained in Section 6.3, except for the change in the compact operator $\widetilde{P}_\gamma$ in (179). However, the convergence rate of the numerical scheme of Section 7 deteriorates drastically, since in this case the kernel K of the operator $\widetilde{P}_\gamma$ in (179) is logrithmically singular (while for the Laplace equation, it is smooth). High-order discretization schemes for such integral equations can be found in the literature (see, for example [1,12,20]).

Needless to say, three-dimensional versions of most problems of mathematical physics are of more immediate applied interest than their two-dimensional versions. Thus, the results of this paper should be viewed as a model for the investigation of the Dirichlet problem for the Laplace equation (or some other elliptic PDE) in three dimensions, with the data specified on an open surface S. When the boundary S is smooth, the transition is fairly straightforward; it becomes more involved when S itself has corners. Both cases are presently under investigation.

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